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Modeling the Solution of an Ordinary Differential Equation by the Functional Voxel Method

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Abstract

This work discusses an approach to modeling an ordinary differential equation by the Functional Voxel method (FV method). The proposed approach is an automated development of the isocline method and is based on the principles of differentiation and integration developed for FV modeling. The isocline method is analyzed as a mean of constructing a tangential field for solving the first and second order ordinary differential equation. The selected examples demonstrate the principle of constructing a FV model as a basis for obtaining integral curves. An algorithm for obtaining an integral curve of a differential equation by the means of the Functional Voxel modeling is described. A visual and numerical comparative analysis of the obtained results of the FV modeling with known examples is carried out. Unlike the isocline method, where the result is a graphical construction of constant tangent lines, in the case of a Functional voxel model we get a graphical representation of the area of local functions at each point of the integral curve corresponding to the solution of the problem.

Keywords: ordinary differential equations, Functional Voxel method (FV method), isocline method, integral curves.

Problem Statement

The main purpose of the research is the fundamental question of the possibility of using Functional-Voxel modeling tools in problems based on obtaining a solution to a differential equation. The implementation of this problem is in demand due to the progressive development of FV-modeling tools in design and control problems, i.e. - it is proposed to model algorithms for solving such problems as: tracing the path with obstacle avoidance [1], algorithms for calculating physical characteristics [2,5], algorithms for solving mathematical programming problems [6], differentiation and integration of the Functional-Voxel model of a function [7], etc.

The isocline method was chosen as the basis of the research not by chance -it is based on the preliminary construction of a tangential field on a given area and the further application of its data in the construction of integral curves. The principle of FV-modeling is similar and requires the preliminary construction of an appropriate FV model to organize an algorithm for solving a problem.

1. The isocline method in solving an ordinary differential equation

A lot of research has been devoted to solving differential equations using various approaches, and each of these approaches has the right to exist and to be applied according to their benefits and advantages [8,9]

One of the graphical methods for solving an ordinary differential equation $y' = f(x,y)$ is based on the construction of isoclines defined as lines along which the value of first derivative is constant [10, 11, 15]. At the same time, automation of this approach requires the generation of a tangential surface in the xOy space and the construction of isolines on it, cut by unidirectional segments (tangents to the integral curve). Figure 1 depicts an example of finding the integral curves for a differential equation $y' = f(x)$, which is often demonstrated in various works:

$$y' = x^2 - x - 2. \tag{1}$$

Here the isoclines are co-directed and parallel to the Oy axis, so it makes no sense to display them. But the segments cutting them with a fixed step, demonstrating the direction of the tangent to the integral curve at a given point, visually represent the general solution of this differential equation. The disadvantage of this graphical approach is the difficulty of using these visual data obtained to build a fairly accurate picture of the integral curves. It is also difficult to contribute such an approach in automated calculations. To do this, it is necessary to express an indefinite integral from the initial differential equation and calculate the corresponding coefficient C, which is a traditional analytical approach.

The lack of analytical expression in the obtained graphical result of the isocline method does not allow the researcher to apply it with confidence in scientific calculations, since it rather carries the visibility of the integral curves' shape and, as a rule, has the substantial loss of accuracy. Since the range of values of the isoclines in this case is a continuous surface, then there exists an integral curve passing through the origin at C=0. Let's conduct a numerical experiment and determine the remaining roots for such an integral curve.

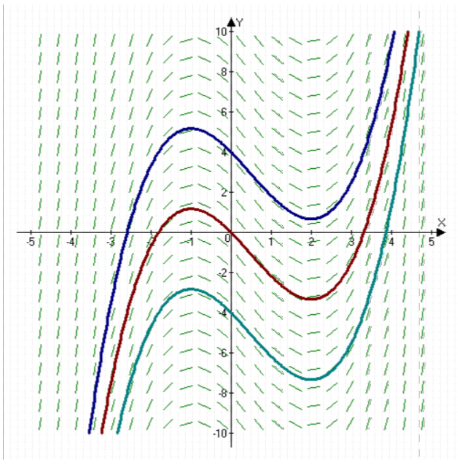


Figure 1 – An Example of the construction of integral curves by isoclines

On integrating the expression (1) we obtain:

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + C. \tag{2}$$

Let's find the roots of the resulting equation (2). Since here we deal with a cubic equation, then, in addition to the origin point, the Ox axis can be intersected twice more. Equating the right part to zero and rearranging we obtain:

$$x(2x^2 - 3x - 12) = 0. \tag{3}$$

Obviously, the first root is $x_1 = 0$ and it remains to solve the quadratic equation in parentheses to get the remaining roots:

$$x_2 = \frac{3 + \sqrt{105}}{4} \approx 3{,}3117377, \qquad x_3 = \frac{3 - \sqrt{105}}{4} \approx -1{,}8117377. \tag{4}$$

In Figure 1, the integral curve (2) shown in red is located in the middle and intersects the $\boldsymbol{Ox}$ axis at coordinates (4). It is clear that the other two integral curves are constructed for $\boldsymbol{C} = \mathbf{4}$ and $\boldsymbol{C} = -\mathbf{4}$.

The functional voxel method (FV-method) [13] provides - on a given area of the analytical function - filling with the local functions describing a linear law for each minimal neighborhood of a point on the area, which makes it possible to apply in further calculations not just a number, but the corresponding analytical expression with all the advantages that follow from this.

Let's try to figure out how, applying the principles of functional voxel modeling, to automate the isocline method for computer application.

2. FV-method for constructing an isoclines' continuous surface

In order to solve equation (1), we will define a certain area (by analogy with the selected example in Figure 1, we will choose the area $x \in [-5; 5], y \in [-5; 5]$).

In the Functional Voxel method, the partial derivative $\partial y/\partial x$ is considered as the ratio $cos\alpha/cos\beta$, where α and β are the angles of deviation of the unit gradient vector from the axes Ox and Oy, respectively. We can say that:

$$y' = x^2 - x - 2 = tg\alpha. \qquad (5)$$

Taking into account the following equalities: $cos\alpha = cos(\pi/2 - arctg\alpha)$ and $cos\beta = sin\alpha = \sqrt{1 - cos\alpha^2}$, we obtain coefficients for the arguments of the tangent equation for the point under consideration, but passing through the origin (local function):

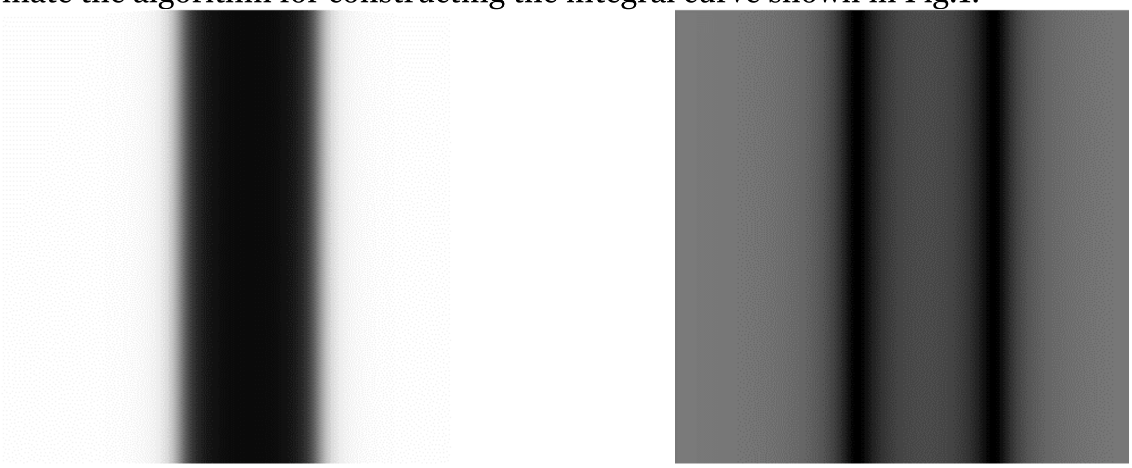
$$cos\alpha x + cos\beta y = 0. \qquad (6)$$

Then we display the specified range of values for each of the cosines as a separate image. A legitimate question arises: why should one tangent surface be converted into two cosine surfaces? In fact, the answer “lies on the surface”. Cosine values are normalized to the interval $[-1; 1]$, which makes it easy to convert them into a color palette $[0 \dots 255]$ which is suitable for computer representation in the form of raster images:

$$Color1 = \frac{(cos\alpha + 1)256}{2} = (cos\alpha + 1)128, \qquad Color2 = (cos\beta + 1)128. \qquad (7)$$

Such information is not only illustrative, but also relatively compact compared to a two-dimensional array of corresponding real values. Additionally, the problem of representing infinitely large values for vertical tangents, etc. disappears. Taking the function $y = f(x)$ as the argument y on the area, we get two raster images responsible for storing $cos\alpha$ and $cos\beta$ on a given area (Fig.2). Further, such images will be called M-images, as is customary in terms of the FV method.

We show that the obtained graphical information in the form of two M-images is sufficient to automate the algorithm for constructing the integral curve shown in Fig.1.



a) b)

Figure 2 – An example of constructing M-images for surfaces: a) $cos\alpha$ и b) $cos\beta$

3. An algorithm for constructing an integral curve using two M-images

In order to organize the sequence of actions of an algorithm for constructing an integral curve using the M-images obtained in Figure 2, it is necessary to determine the relation between the dimensions of the real area of the function specified by real numbers and the M-image resolution, which is integer and contain information about the number of points in two directions of the raster.

Following the problem discussed earlier, the function area is set by the parameters:

$$X_{min} = -5, X_{max} = 5, Y_{min} = -5, Y_{max} = 5.$$

The dimensions of the M-image are $X_{bmp} = 400$, $Y_{bmp} = 400$. Let's determine the scaling factor along the axes, which ensures the transition from the function area to the image and back in the calculations:

$$K_x = \frac{X_{max} - X_{min}}{X_{bmp}}, K_y = \frac{Y_{max} - Y_{min}}{Y_{bmp}}. \tag{8}$$

Since the M-image contains 400 points along the Ox axis, and the origin for the region is located in the middle, and there is also information that the desired integral curve passes through the origin, then it is proposed to build such an integral curve first in the positive direction of the semi-axis from the point *(0,0)*, and then in the negative direction.

Let's set the starting point to the origin, recalculating it to the M-image coordinates:

$$X = \frac{(x - X_{min})}{K_x}, \qquad Y = \frac{(y - Y_{min})}{K_y}. \tag{9}$$

It is obvious that at coordinates *(0,0)* on the M-image, the point will be in the middle of the window with coordinates *(200, 200)*. For the resulting point, there is a specific color on both M-images ($Color1$ и $Color2$),which, when converted back, becomes the cosine value:

$$cos\alpha = \frac{2Color1 - 256}{256}, \qquad cos\beta = \frac{2Color2 - 256}{256}. \tag{10}$$

To obtain a local equation at a point with coordinates *(0,0)*, we find the third component of the gradient of the neighborhood of this point:

$$cos\gamma = -cos\alpha x - cos\beta y. \tag{11}$$

In the case of the first point, when both coordinates are zero $cos\gamma = 0$, which means that the line described by this local function really passes through the origin.

To determine the value of the next point of the integral curve, we perform a shift along the Ox axis by a step K_x:

$$y' = -\frac{cos\alpha}{cos\beta}(x + K_x) - \frac{cos\gamma}{cos\beta}. \tag{12}$$

Having obtained the solution of the integral curve at the next point, we proceed to it:

$$x = x + K_x, \qquad y = y'. \tag{13}$$

Now we calculate the X, Y coordinates again using the formula (9), we get the color $Color1, Color2$ on two M-images at a new point (X, Y) in order to determine the new $cos\alpha$ and $cos\beta$ for it using the formula (10). We define $cos\gamma$ for the obtained point using the formula (11) and proceed to the finding a new point of the integral curve (12, 13) and then repeat the process until one of the coordinates reaches the boundary of the region.

Similarly, the algorithm is built in the opposite direction from a given point, with the only difference that the $cos\gamma$ parameter is determined for the point $(x - K_x)$, since we assume that the current point also belongs to the previous neighborhood.

Figure 3 shows a bundle of integral curves for *C=[-3,-1,-2,0,1,2,3]*, superimposed on the M-image with $cos\alpha$ value mapping for clarity.

On the resulting integral curve, it can be seen that the roots of its function at C=0 correspond to the required values (4).

Consider the example $y' = f(x,y)$, where in addition to the argument x, the function y itself is also present in the equation. To do this, we will also choose one of the cases often considered in classical textbooks [16]:

$$y' = 2x - y. \quad (14)$$

An image with the construction of isoclines and the resulting integral curves, as well as the automatic construction of an integral curve by the FV method is shown in Figure 4. It is not difficult to make sure that the integral curve passing through the origin has a single root at the extreme point. We equate the derivative to zero to construct the isocline of extremes:

$$y' = 2x - y = 0 \text{ или } y = 2x. \quad (15)$$

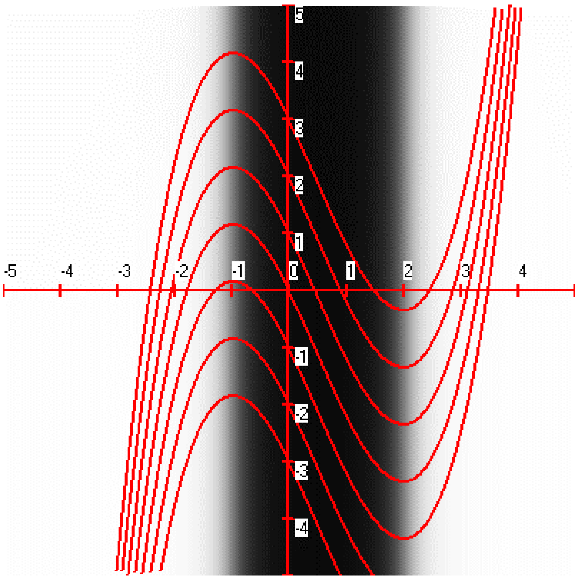
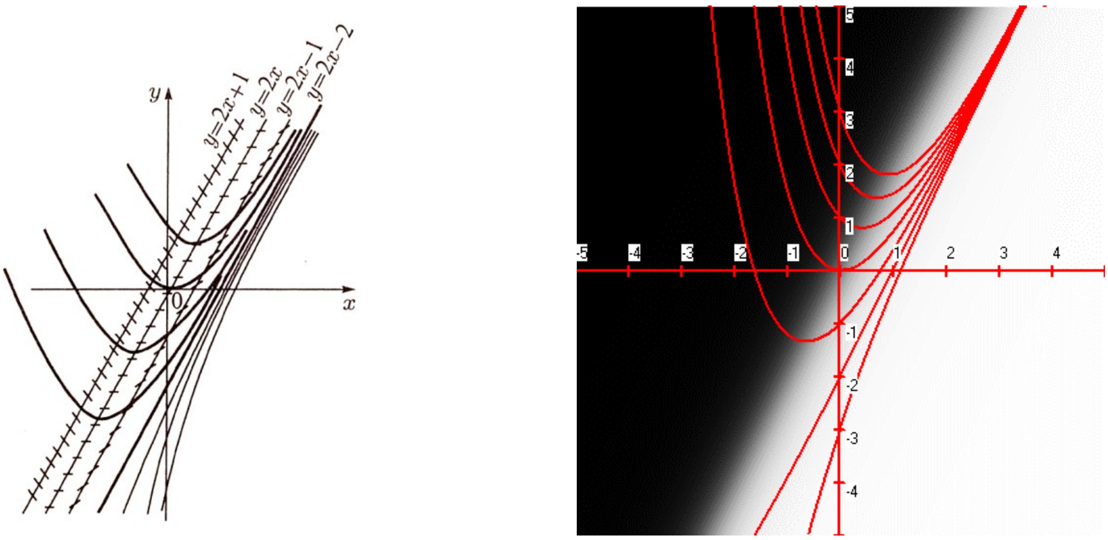


Figure 3 – The result of the FV-constructing an integral curve for (1)

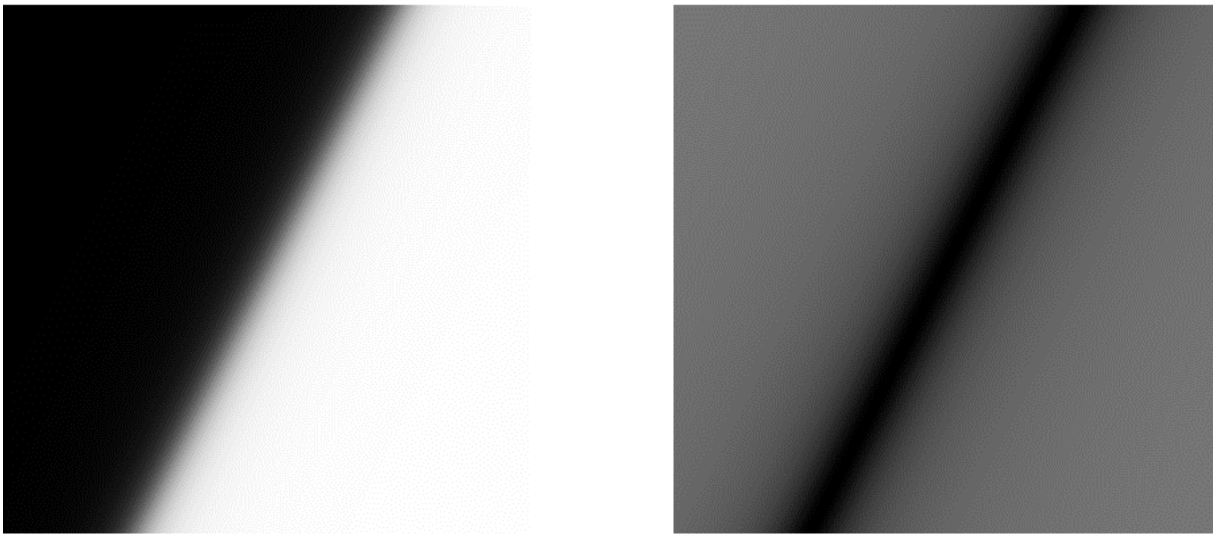
The resulting straight line of the isocline passes through the origin, which means that the point of the extremum of the integral curve for C=0 will be located at the origin. We will set the starting point for the algorithm to work right there and start the process of constructing the integral curve. The result confirming the correctness of the algorithm is shown in Figure 4a. In Figure 4b, the integral curve passing through the origin is shown in red.



a b

Figure 4 – The result of the FV-constructing integral curves for (14): a) the isocline method, b) FV-method

For the algorithm to work, it is necessary to pre-construct M-images of the mapping on a given area of $cos\alpha$ and $cos\beta$ as shown in Figure 5.



$cos\alpha$ $cos\beta$

Figure 5 – mapping the isocline region as a FV-model for the equation (1.14)

As the next example, it is proposed to consider the quadratic expression of the differential equation:

$$y' = y - x^2 + 2x - 2. \quad (16)$$

The result of the construction of isoclines and obtaining the integral curves in the traditional version is shown in Figure 6.

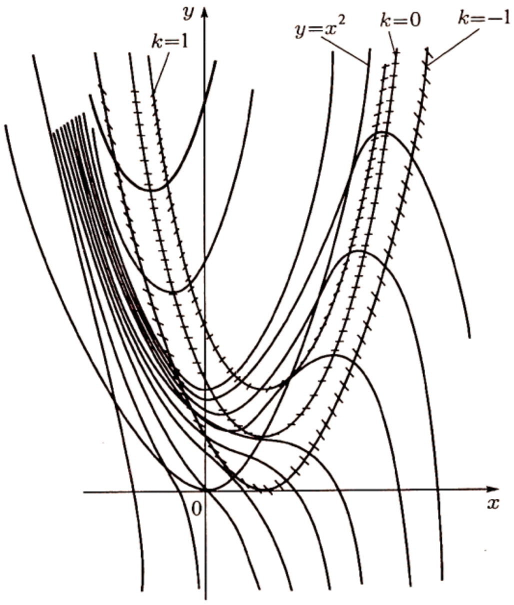


Figure 6 – The construction of integral curves of the equation (16) by the isocline method

Let's visually highlight for ourselves in Figure 6 the integral curve passing through the origin. For the algorithm to work, we define the corresponding M-images describing a given area of the tangential field decomposed into cosines. The result of constructing such M-images is shown in Figure 7. Figure 8 demonstrates the solution of the problem by the FV-method.

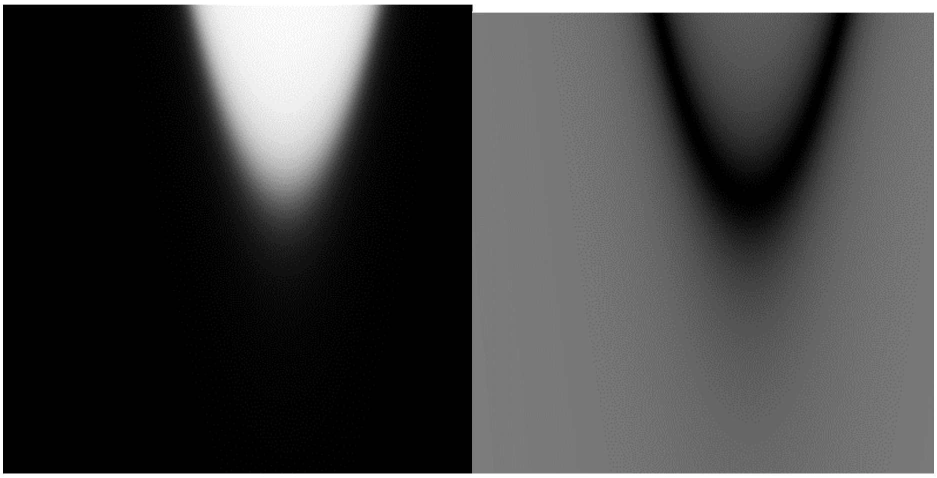


Figure 7 – Representation of the isocline region in the form of a FV model for the equation (16)

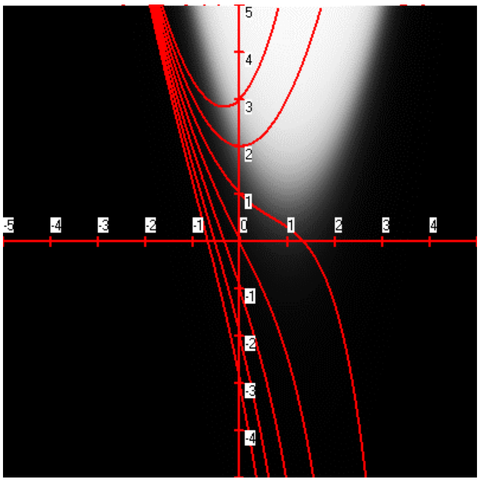


Figure 8 – Integral curves of the equation (16)

Let's consider the last test example of a first-order differential equation involving a periodic function:

$$y' = \sin(x + y). \quad (17)$$

We will compare the result with the traditional image included in many textbooks (Fig.9) [15].

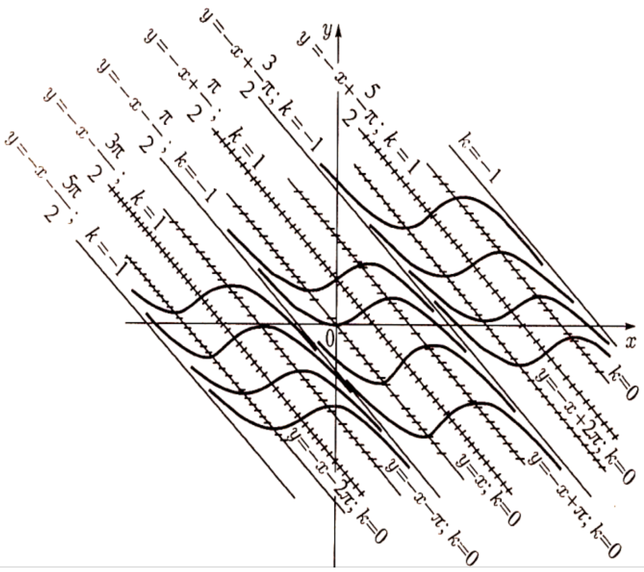


Figure 9 – Construction of integral curves of the equation (17)

Let's construct M-images to solve equation (17) (Fig. 10) and apply the proposed algorithm for constructing an integral curve to these two M-images. The result of constructing integral curves for equation (17) is shown in Figure 11.

The conducted research shows that the application of Functional Voxel modeling to solving first-order ordinary differential equations makes it possible to transfer the graphical isocline method to an automated basis, with the only difference that the information field formed in this case does not display the tangent value, but is based on information about the components of the unit gradient vector in the investigated area of the differential equation.

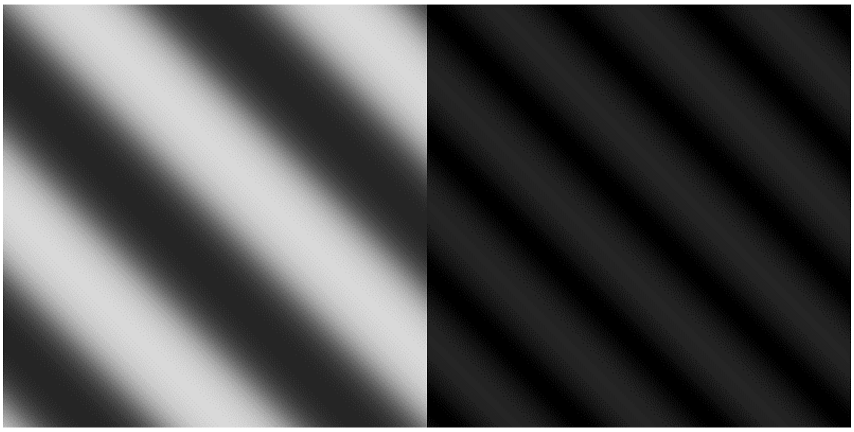


Figure 10 – Representation of the isocline region in the form of a FV model for the equation (17)

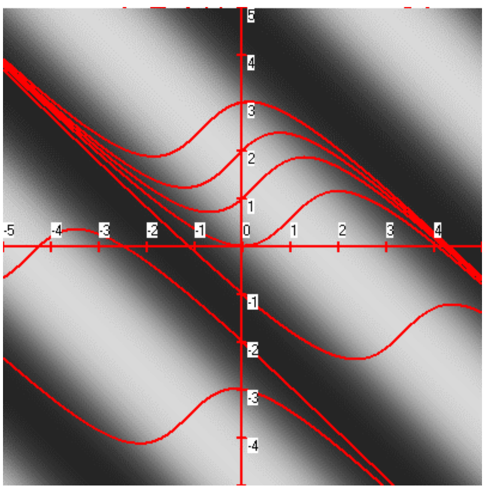


Figure 11 – Construction of an integral curve for equation (17) passing through the origin

4. The solution of the second order ordinary integral equation of by the isocline method

Since the isocline method is also applied to solving some second-order equations, we will demonstrate the solution of an example taken from [14] which is displayed at (Fig.12) for comparison. Consider the equations in a reduced form:

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}, x\right) = 0. \tag{18}$$

A new variable $v = dx/dt$ is introduced. Then we have:

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} \tag{19}$$

and equation (18) takes the form of a first-order equation:

$$\frac{dv}{dx} = \frac{f(v,x)}{v}. \tag{20}$$

Let's consider an example of solving a differential equation of the form:

$$\frac{d^2x}{dt^2} + x = 0. \tag{21}$$

We assume that $dx/dt = v$. Then equation (21) takes the form:

$$v\frac{dv}{dx} + x = 0, \qquad \text{или} \qquad \frac{dv}{dx} = -\frac{x}{v}. \tag{22}$$

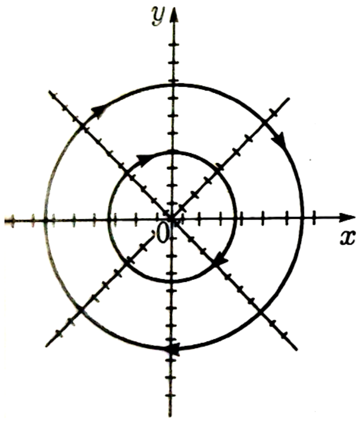
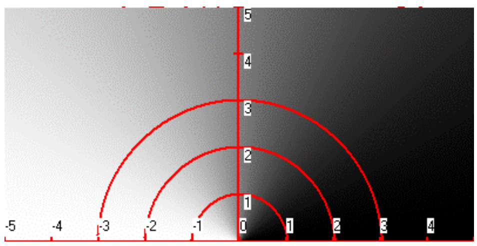


Figure 12 – Construction of integral curves by the isocline method for the equation (19)

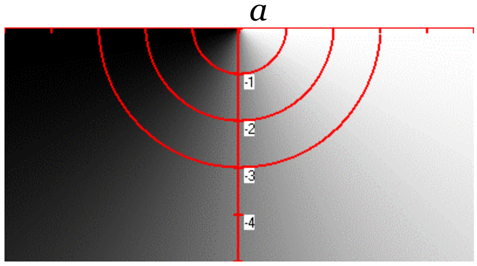
We apply the Functional Voxel approach to solving equation (19). Let us pay attention to the fact that in all examples of such a reduction of the initial differential equation to the first order, division by the function v is implied, which leads along the Oy axis to a discontinuity of the continuous surface (Fig.13), since in order to obtain M-images, the function (22) is transformed into a function:

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{23}$$

To construct integral curves, we use the regions $x \in [-5; 5], y \in [-5; 0)$ and $x \in [-5; 5], y \in (0; 5]$ alternately, i.e. we divide the domain of definition of a function into two subdomains before and after the Oy axis. Figure 13 *a* and *b* show the result of constructing integral curves at *C*=1, *C*=2 and *C*=3 for the positive area, and at *C*=-1, *C*=-2 and *C*=-3 for the negative.



a



b

Figure 13 – Construction of integral curves by the FV method for the equation (19): a) the solution in area $y \in (0; 5]$, b) the solution in area $y \in [-5; 0)$

It is not difficult to combine these two images in the further calculations to obtain the comprehensive picture.

Conclusions

The conducted research has shown the possibility of using the Functional Voxel modeling in solving ordinary differential equations of the form $y' = f(x, y)$. The solution of the equation of the form $y'' = f(x, y', y)$ is demonstrated. In the future, it is proposed to consider the use of the FV modeling in solving applied problems based on the application of the first and second order ordinary differential equations, the development of the principles for the algorithm for ordinary differential equations of the form: $z' = f(x, y, z)$ and $z'' = f(x, y, z, z')$. Since the Functional Voxel model allows us to analytically describe complex geometry in a given area, then its application in problems of fluid or gas motion as a basis for modeling the differential laws formed in this case seems to the authors very promising and relevant.

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References

1. Vassilyev, S.N., Loktev, M.A., Tolok, A.V., Tolok, N.B., and Ulyanov, S.A., Route planning in 3D environment with a multivariant model, SPIIRAS Proc., 2016. no. 2(45), p. 5–25.
2. A.M. Plaksin, S.A. Pushkarev. Geometric modeling of thermal characteristics of objects using the functional-voxel method // Geometry and Graphics. 2020. V. 8, № 1. p. 25-32.
3. A.V. Tolok, M.A. Loktev, N.B. Tolok, A.M. Plaksin, S.A. Pushkarev. Visual diagnostics of physical quantities based on the functional-voxel modeling method (2020). Scientific Visualization 12.3: 51 - 60, DOI: 10.26583/sv.12.3.05
4. Alexey Tolok, Natalya Tolok. Arithmetic in Functional-Voxel Modeling (2022). Scientific Visualization 14.3: 107 - 121, DOI: 10.26583/sv.14.3.08
5. Plaksin A.M., Pushkarev S.A., Sycheva A.A., Harlanova P.M. Geometric modeling of stress visualization tools based on the functional voxel method // Geometry and Graphics. 2020. V. 8, № 3. p. 36-43.
6. Tolok A,V, Tolok N.B. Mathematical Programming Problems Solving by Functional Voxel Method // Automation and Remote Control. 2018. Vol. 79 (72), Vol. 9 (No.12). p. 1703-1712 (345-354).
7. Tolok A,V, Tolok N.B. Differentiation and Integration in Functional Voxel Modeling // CONTROL SCIENCES. 2022. № 5. p. 51-57.
8. Soundararajan, R.; Subburayan, V.; Wong, P. J.Y. Streamline Diffusion Finite Element Method for Singularly Perturbed 1DParabolic Convection Diffusion Differential Equations with Line Discontinuous Source / Mathematics. – 2023. – Vol. 11. – Art. no. 2034. – DOI: https://doi.org/ 10.3390/math11092034.
9. Mohammed, M. Well-Posedness for Nonlinear Parabolic Stochastic Differential Equations with Nonlinear Robin Conditions / Symmetry. – 2022. – Vol. 18. – Art. no. 1722.
10. Egorov A.I. Ordinary differential equations with applications. M.: Phismatlit, 2003. 384 pp.
11. Romanenko V.K. The course of differential equations and calculus of variations. M.: Laboratory of basic knowledge, 2000. 344 pp.
12. Stepanov V.V. The course of differential equations. M.: Editorial URSS, 2004. 472 pp.
13. A.V. Tolok, N.B. Tolok. Functional-Voxel Modeling of The Cauchy Problem (2024). Scientific Visualization 16.1: 105 - 111, DOI: 10.26583/sv.16.1.09
14. Konev V.V. Partial equations: Tutorial / Tomsk Polytechnic University. – https://portal.tpu.ru/SHARED/k/KONVAL/notes/Partial.pdf.
15. Krasnov M.L., Kiselyov A.I., Makarenko G.I. Ordinary differential equations: Problems and examples with detailed solutions: Subject guide. Stereotype. M.: Lenand, 2019. – 256 pp. (All Higher Mathematics in problems)