

Possibility of Using the Gershberg-Papoulis Method in the Problem of Phase Structure Reconstructing from Low-angle Hilbertograms

E.V. Arbuzov^{1,A,B}, V.A. Arbuzov^{2,A}, Yu.N. Dubnishchev^{3,A}, O.S. Zolotukhina^{4,A},
M.M. Lapikov^{5,A}, V.V. Lukashov^{6,A}

^A Kutateladze Institute of Thermophysics of the Siberian Branch of the Russian Academy of Sciences

^B Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences

¹ ORCID: 0000-0001-9488-8650, arbuzov@math.nsc.ru

² ORCID: 0000-0003-2404-326X, arbuzov@itp.nsc.ru

³ ORCID: 0000-0001-7874-039X, dubnistchev@itp.nsc.ru

⁴ ORCID: 0000-0003-3486-4459, melexina-olga17@ya.ru

⁵ ORCID: 0009-0009-0759-8627, mlapikov1@list.ru

⁶ ORCID: 0000-0001-8178-7607, luka@itp.nsc.ru

Abstract

The possibility of processing small-view hilbertograms by the Gershberg-Papulis method to restore the refractive index of phase objects is discussed. The method consists in iterative transitions from estimating a function in the Fourier plane to estimating it in a coordinate space with an adjustment using a priori information. The spectrum of the function is determined on the entire frequency plane as an iterative process result. Numerical simulation of the refractive index reconstruction for various test functions was performed using the Gershberg-Papulis method using Radon data known for four angles. Experimental studies on the Hilbert diagnostics example of reacting media (flames) in a high-speed shooting mode (up to 2000 frames per second) were performed using a four-angle tomographic complex implemented on the basis of an upgraded IAB-463M shadow device.

Keywords: Hilbert optics, optical tomography, the Gershberg-Papoulis method.

1. Introduction

The study of phase objects using spatial optical filtering methods remains an urgent task to this day [1, 2]. Methods for observing phase objects are historically called shadow methods (SM), and the corresponding devices are called shadow devices [3, 4]. SMs have taken their rightful place as popular and effective techniques in the diagnosis of transparent inhomogeneities in various media [5]. However, SMs have not yet received the recognized status of accurate and strictly quantitative methods of phase measurements.

Monographs [6, 7] were published in 2003 and 2007, which present modern methods for studying phase objects based on the analysis of spatio-temporal and frequency parameters of optical signals. These are laser Doppler anemometry and Hilbert filtering of light fields disturbed by the medium under study. Hilbert transformations made it possible to present the theory of classical shadow devices in the form of a compact mathematical apparatus and to improve the technique for diagnosing signals in these optical systems. Hilbert optics methods can provide higher sensitivity, since the structure of “hilbertograms” contains features that make it possible to obtain numerical characteristics of the optical density field without using

interferometry for these purposes. Work [8] is one of the first to highlight the Hilbert transform usefulness in optical Fourier transform spectroscopy.

The measuring results of the temperature fields of gaseous and reacting media (flames) and the molar concentrations of fuel combustion products using Hilbert optics methods in the approximation of visualization object axial symmetry are presented in [9, 10]. Diagnosis of complex (asymmetrical) structures using computed tomography (CT) methods [11, 12] is the next stage in the research development. The problem associated with the organizing difficulty of a sufficient number of projections or the small angular range of diagnosing an object is one of the optical CT key problems.

The Gershberg-Papulis algorithm is one of the common methods for solving the problem of reconstructing the structures under study in low-angle tomography [13–15]. It is based on an iterative process that allows you to refine the results, minimizing the discrepancies between experimental data and theoretical estimates at each iteration.

The purpose of this work is to study the Gershberg-Papulis method and its application for the reconstruction of the phase optical density fields of gaseous, condensed and reacting (flames) media according to low-angle Hilbert tomography data.

2. Low-angle Hilbert tomography

The optical complex was developed to implement Hilbert tomography based on the upgraded shadow device [9] IAB-463M, which allows probing the object under study from four angles and synchronously recording tomographic projections of the visualized phase structures with one video camera (Fig. 1).

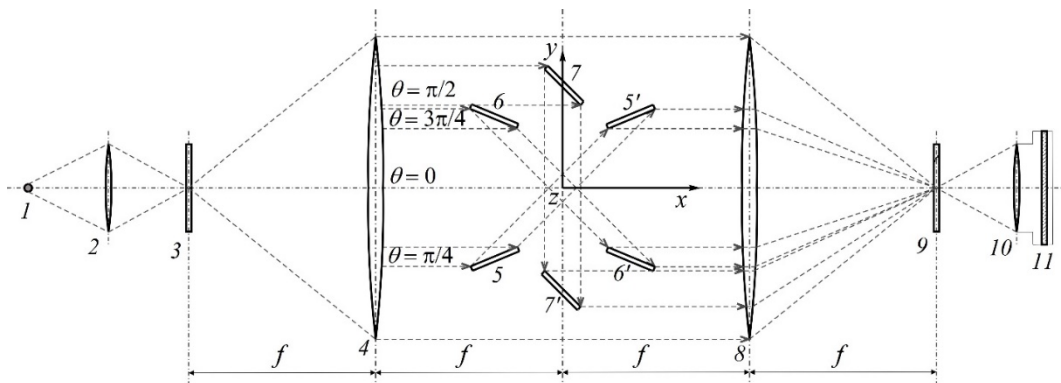


Fig. 1. Simplified diagram of the Hilbert tomograph.

The complex contains an illumination module consisting of a radiation source 1, the objective 2, and a slit diaphragm 3 placed in the front Fourier plane of the objective 4. A KLM-532-2000 laser with spatial coherence suppression is used as a radiation source. The structure of the probing light fields that implement 4R tomographic diagnostics is formed by a pair configuration of mirrors 5 and 5', 6 and 6', 7 and 7', forming beams oriented relative to the optical axis of the shadow device at angles $\theta_p = \pi(p - 1)/4$, where the projection number is $p = 1, \dots, 4$. Overall dimensions of the mirrors are $100 \times 15 \times 145$ mm.

The Fourier spectrum of phase perturbations induced in probing fields by the object of study is localized in the frequency plane of objective 8, where a quadrant phase Hilbert filter 9 is placed [7]. The lens 10 of the high-speed video camera converts the filtered field, depending on the spectral characteristics of the light source, into analytical or Hilbert-coupled optical signals, which are recorded on the CCD matrix 11. The choice of the tomograph technical solution is due to the possibility of using a large field of view (400 mm) of the IAB-463M shadow device.

A linear light source oriented along one of the spatial-frequency axes, in combination with a quadrant phase Hilbert filter, provides a one-dimensional Hilbert transform of the p -tomographic component of the optical field:

$$I_p(y_p, z) = \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s_p(y', z)}{y_p - y'} dy' \right|^2 = |\hat{s}_p(y_p, z)|^2,$$

where $I_p(y_p, z)$ is the signal intensity recorded by the video camera, $\hat{s}_p(y_p, z)$ is the signal Hilbert image $s_p(y_p, z) = e^{i\varphi_p(y_p, z)}$, $\varphi_p(y_p, z)$ is the phase function, which is determined by the Radon transform of the refractive index $n(x_p, y_p, z)$ in the local structure of the medium under study:

$$\varphi_p(y_p, z) = \frac{2\pi}{\lambda} \int_{x'}^{x''} [n(x_p, y_p, z) - n_\infty] dx_p,$$

λ is the wavelength, n_∞ is the air refractive index, x' and x'' are the entry and exit points of the beam relative to the local structure of the medium under study for a certain tomographic component.

Experiments to study combustion processes were carried out using a Hilbert tomographic complex. Examples of hilbertograms of a burner and candle flame are shown in Fig. 2 and 3.

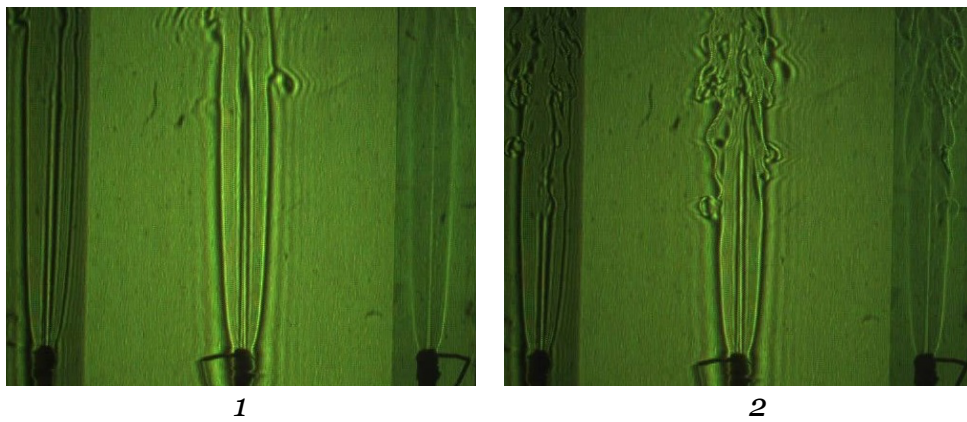


Fig. 2. 3R burner flame hilbertograms: the projection $\theta_p = 3\pi/4$, the frontal projection $\theta_p = 0$ and the angled projection $\theta_p = \pi/4$ are shown sequentially.

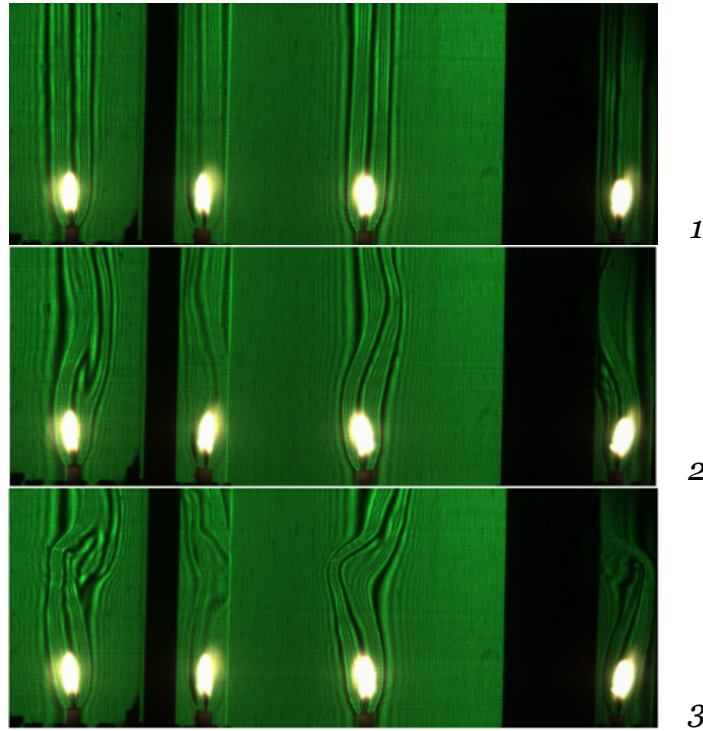


Fig. 3. 4R candle flame hilbertograms: projection $\theta_p = \pi/2$, $\theta_p = 3\pi/4$, front projection $\theta_p = 0$ and angle projection $\theta_p = \pi/4$.

3. The Gershberg-Papoulis method

The tomography inverse problem in a parallel setting is to restore the refractive index function $n = n(x, y)$ from the integrals values of it along the straight lines $L: \vec{r} \cdot \vec{e}_\theta = s$, $\vec{r} = (x, y)$ and $\vec{e}_\theta = (\cos\theta, \sin\theta)$:

$$\mathcal{R}_\theta n(s) = \int_L n(x, y) dl = \iint_{R^2} \delta(s - \vec{r} \cdot \vec{e}_\theta) n(x, y) dx dy.$$

Many inversion formulas for the Radon problem are based on the central section theorem [16], which states that the (one-dimensional) Fourier transform of the projection $\mathcal{R}_\theta n(s)$ is equal to the cross section of the two-dimensional Fourier transform of the function $n = n(x, y)$:

$$\widehat{\mathcal{R}_\theta n}(\omega) = F[n](W_\theta), \quad W_\theta = (\omega \cos\theta, \omega \sin\theta).$$

This theorem underlies many algorithms of computed tomography, in particular, the back projection method [15].

The Gershberg-Papoulis algorithm is one of the most efficient methods for reconstructing functions from their Radon projections, especially in cases where the scanning directions number is small or their angular range is limited. The method essence lies in the fact that a priori information about the desired function $n(x, y)$ and its known projections $\mathcal{R}_\theta n(s)$ is used to create an initial approximation and subsequent correction in the coordinate and frequency spaces. The non-negativity and finiteness of the desired function are used as a priori information:

$$n(x, y) \geq 0; \quad n(x, y) = 0; \quad x^2 + y^2 \geq 1.$$

The function values $N(W) = F[n](W)$ in the directions along the vectors $\vec{e}_p = (\cos\theta_p, \sin\theta_p)$ are known in the Fourier plane from the Radon data. Thus, the Fourier transform of the desired function $n(x, y)$ is known on the set M :

$$M = \{W = \omega \vec{e}_p, -\infty < \omega < \infty\}.$$

If H_M is the characteristic function of the set M :

$$H_M = \begin{cases} 1, & W \in M \\ 0, & W \notin M \end{cases};$$

then the values are known

$$S_M(W) = H_M(W)N(W).$$

The following operations must be performed to reconstruct the refractive index function:

1. One-dimensional Fourier transforms are computed from known Radon data. Thus, the function S_M is determined, equal to the values of the two-dimensional Fourier transform of the desired function in the directions corresponding to the projections angles, and equal to zero at Fourier plane other points.

2. The initial approximation n^0 is determined: the inverse two-dimensional Fourier transform of the S_M function is performed. A priori information about the refractive index positive-ness n and the area boundedness of its assignment is introduced (the C_M operator is applied).

3. A two-dimensional Fourier transform is performed from the initial approximation. The spectrum values in the directions corresponding to the projection angles are replaced by the values calculated in step 1.

4. The inverse two-dimensional Fourier transform of the function obtained at the previous step is performed and the C_M operator is applied to the result.

5. The criterion for the iterative process end is checked (if it is not fulfilled, then steps 3, 4 are repeated): the norm smallness of obtained tomogram deviation from its estimate at the previous stage Δ_m :

$$\Delta_m^2 = \frac{\sum_i \sum_j (n_{i,j}^m - n_{i,j}^{m+1})^2}{\sum_i \sum_j (n_{i,j}^m)^2}.$$

As a result, the reconstruction algorithm can be represented as

$$N^0 = S_M, \quad n^0 = C_M F^{-1}[N^0], \\ N^{m+1} = S_M + F^{+1}[n^m](1 - H_M), \quad n^{m+1} = C_M F^{-1}[N^{m+1}].$$

The use of the Gershberg-Papulis method for the refractive index reconstruction from hiltograms is due to the following. The projection values for all selected observation angles are determined at those points that correspond to the Radon integrals along the straight lines passing through the nodes of the sampling grid when obtaining a digital image using a system of mirrors. The sampling grid nodes correspond to the photomatrix resolution. Therefore, the discrete analog of the central section theorem allows one to obtain initial data in the Fourier plane without preliminary interpolation.

4. Test objects reconstruction

Linear combinations of the following functions were considered to evaluate the application effectiveness of the Hershberg-Papulis method of recovering an unknown function from its Radon data obtained from four projections:

$$f(x, y) = A e^{1/a^2} e^{-1/|a-|\vec{r}-\vec{r}_0||^2} \text{Heaviside}(a - |\vec{r} - \vec{r}_0|),$$

$$g(x, y) = A e^{-|\vec{r}-\vec{r}_0|^2/s^2},$$

$$h(x, y) = A \text{Heaviside}(a - |\vec{r} - \vec{r}_0|),$$

where

$$\vec{r} = (x, y),$$

$$r = \sqrt{x^2 + y^2},$$

$$\vec{r}_0 = (x_0, y_0),$$

$$r_0 = \sqrt{x_0^2 + y_0^2}.$$

The reconstruction results are shown in Fig. 3–9, where (a) is the original function, (b) is the reconstructed function by the Hershberg-Papulis method with indication of the root-mean-square recovery error Δ_m , (c) is the reconstructed function by the back projection method with filtering, (d) is the reconstructed function by the method of back projections without filtering.

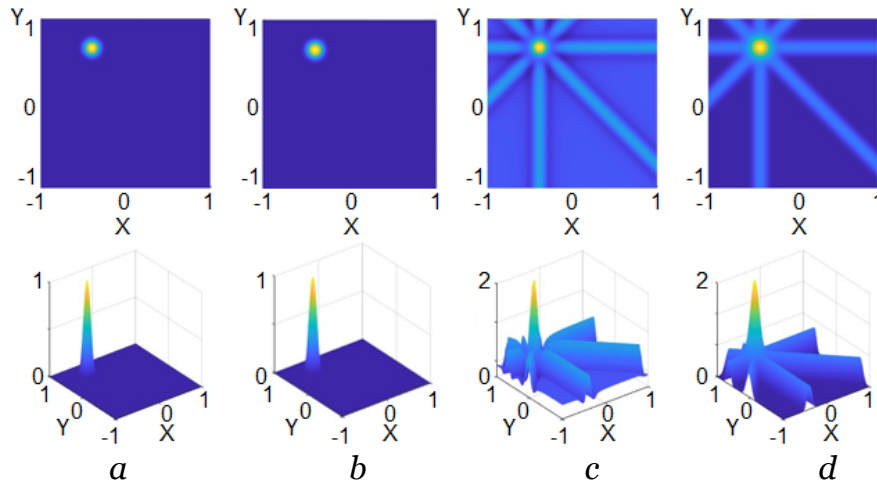


Fig. 4. $n(x, y) = f_p(x, y)$; $x_0 = -0,4$; $y_0 = 0,65$; $A = 1$; $a = 0,3$;
 $\Delta_m = 1,53\%$, number of iterations $m = 751$.

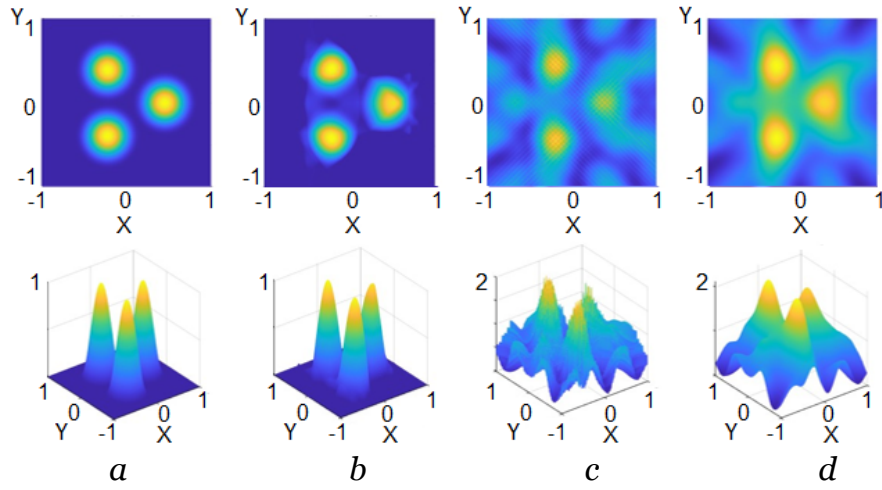


Fig. 5. $(x, y) = \sum_{p=1}^3 f_p(x, y)$;
 $x_0 = 0,45 (1; -0,5; -0,5)$; $y_0 = 0,45 (0; 0,86; -0,86)$;
 $A = (1; 1; 1)$; $a = (0,5; 0,5; 0,5)$;
 $\Delta_m = 11,53\%$, number of iterations $m = 840$.

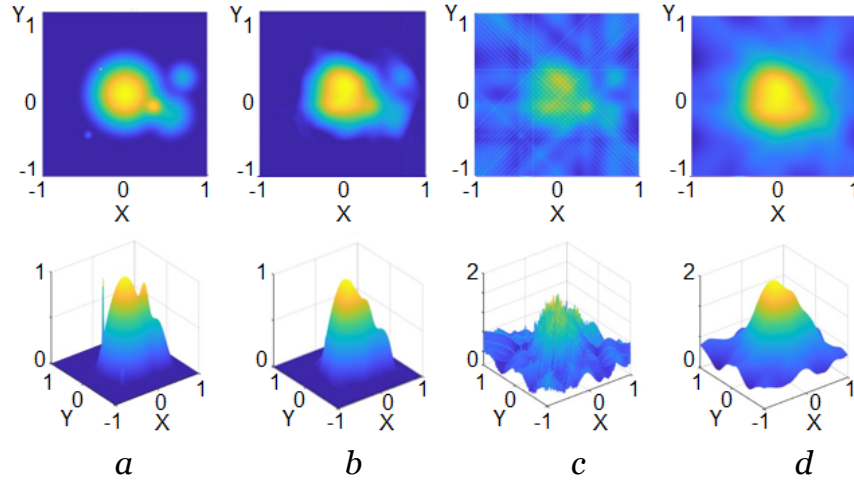


Fig. 6. $(x, y) = \sum_{p=1}^6 f_p(x, y)$;
 $x_0 = (0; 0,35; 0,55; 0,7; -0,45; -0,3)$; $y_0 = (0; -0,15; -0,25; 0,2; -0,5; 0,3)$;
 $A = (1; 0,3; 0,5; 0,4; 0,2; 0,8)$; $a = (0,7; 0,3; 0,5; 0,4; 0,2; 0,1)$;
 $\Delta_m = 9,99\%$, number of iterations $m = 96$.

Further, functions similar to the test examples presented in [17] are considered.

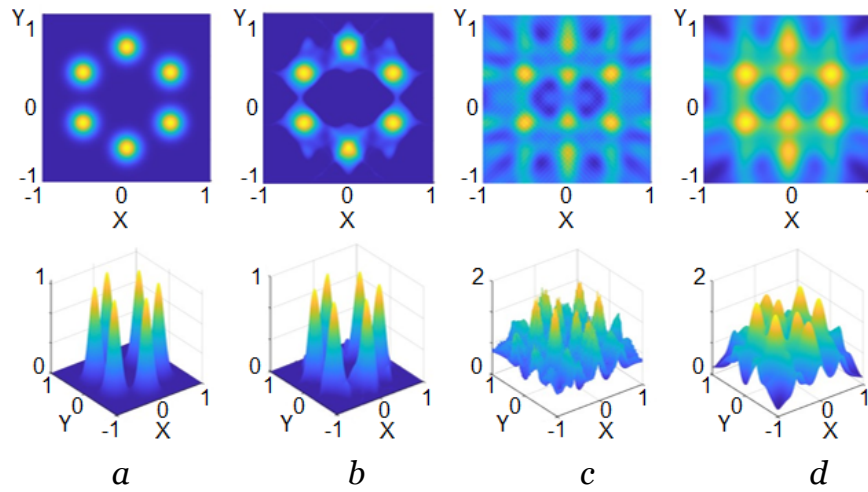


Fig. 7. $n(x, y) = \sum_{k=1}^6 g_k(x, y)$;
 $x_0 = 0,65 (0,86; 0; -0,86; -0,86; 0; 0,86)$; $y_0 = 0,65 (0,5; 1; 0,5; -0,5; -1; -0,5)$;
 $A = (1; 1; 1; 1; 1; 1)$; $s = (0,1; 0,1; 0,1; 0,1; 0,1; 0,1)$;
 $\Delta_m = 18,44\%$, number of iterations $m = 573$.

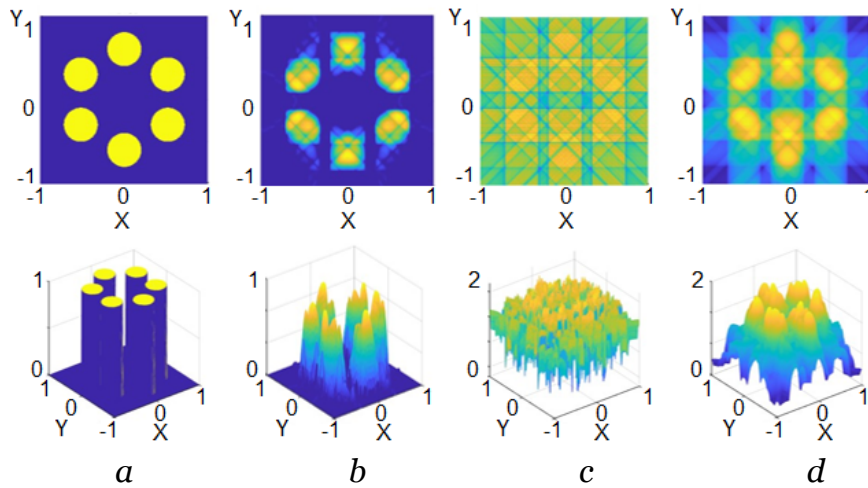


Fig. 8. $n(x, y) = \sum_{p=1}^6 h_p(x, y; a_p)$;
 $x_0 = 0,65 (0,86; 0; -0,86; -0,86; 0; 0,86)$; $y_0 = 0,65 (0,5; 1; 0,5; -0,5; -1; -0,5)$;
 $A = (1; 1; 1; 1; 1; 1)$; $a = (0,2; 0,2; 0,2; 0,2; 0,2; 0,2)$;
 $\Delta_m = 34,85\%$, number of iterations $m = 302$.

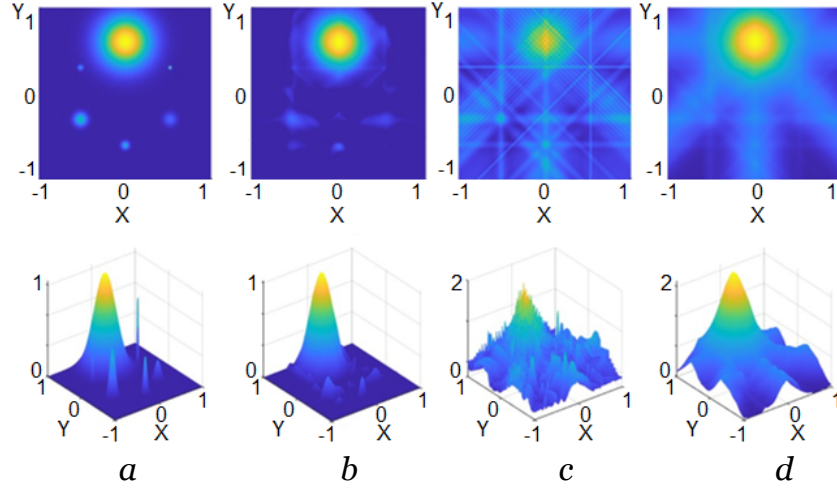


Fig. 9. $n(x, y) = \sum_{p=1}^6 f_p(x, y; a_p)$;
 $x_0 = 0,65 (0,86; 0; -0,86; -0,86; 0; 0,86)$; $y_0 = 0,65 (0,5; 1; 0,5; -0,5; -1; -0,5)$;
 $A = (0,8; 1; 0,3; 0,5; 0,4; 0,2)$; $a = (0,01; 0,2; 0,02; 0,05; 0,03; 0,05)$;
 $\Delta_m = 10,92\%$, number of iterations $m = 404$.

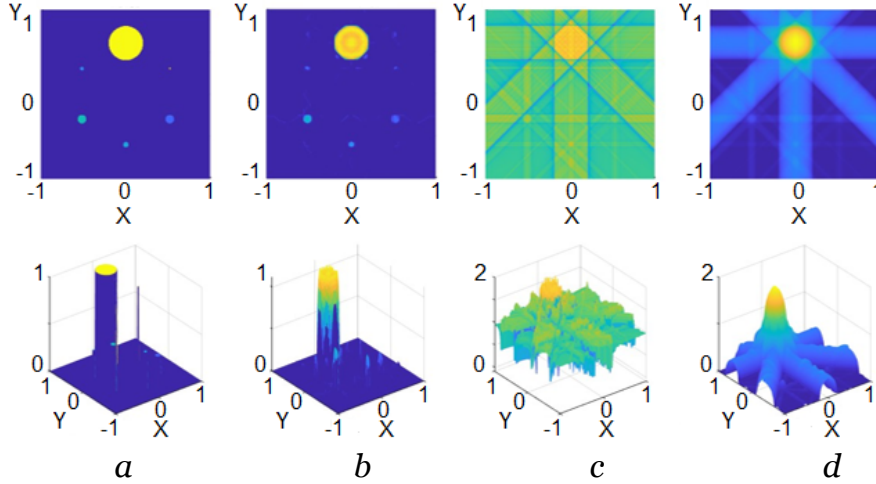


Fig. 10. $n(x, y) = \sum_{p=1}^6 h_p(x, y; a_p)$;
 $x_0 = 0,65 (0,86; 0; -0,86; -0,86; 0; 0,86)$; $y_0 = 0,65 (0,5; 1; 0,5; -0,5; -1; -0,5)$;
 $A = (0,8; 1; 0,3; 0,5; 0,4; 0,2)$; $a = (0,01; 0,2; 0,02; 0,05; 0,03; 0,05)$;
 $\Delta_m = 14,66\%$, number of iterations $m = 1185$.

Since the Gershberg-Papoulis iterative method interpolates the reconstructed function spectrum by known values given on the straight lines corresponding to the scanning directions, the reconstruction result will be better if the original function Fourier spectrum is localized in the low spatial frequency region. The given examples of model functions are an illustration of this provision.

5. Conclusion

The possibility of using the Gershberg-Papoulis method for solving the low-angle optical Hilbert diagnostics problem is investigated in this work. Numerical simulation was performed to evaluate the method effectiveness.

The Hershberg-Papoulis method is an important tool in the field of computed tomography and optics. It solves the problems associated with the image restoration with limited projections, and finds application in various fields of science and industry. However, it may be necessary to modify this method or use additional methods for more accurate image restoration.

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