A Study of the Algorithm for Constructing a Second-Order Surface Defined by Nine Points

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Abstract

The paper presents the experimental implementation of the historical problem of constructing a quadric defined by nine points. It considers the well-known Engel algorithm (Engel J. H., 1889). Until recently, the algorithm has shown only the theoretical potential of constructing a quadric. The paper describes the software used for the experimental implementation and exploration of the algorithm. The programs are written in the AutoLisp language. The experiments were carried out in the AutoCAD suite.

This study presents geometric constructions resulting in the implementation of the problem. The construction stages are considered in detail. The first and main stage defines nine points to constructing a conic on the quadric surface. The next stages define the quadric determinant of three conics and construct the center and the principal axes of the quadric, the frame, and the final surface. The paper studies the construction features at all stages. The primary focus is to identify and study the inner relationships of parameters leading to the appearance of real or imaginary solutions.

The scientific novelty of the paper lies in determining the conditions, under which the solution is reached without operations using imaginary parameters. The region of exact real solutions for a set of nine points is determined. The paper shows that this region has the form of a curved line approximated by a hyperbola and proposes its construction algorithm.

The author proposes and investigates an algorithm for constructing the principal axes of a quadric, consisting in determining a cutting plane for which the foot of the perpendicular dropped from the center of the quadric coincides with the center of the conic section. In this case, the perpendicular takes the position of one of the principal axes of the quadric.

The relevance of the paper lies in the experimental implementation of the historical Engel algorithm, as well as the development and demonstration of modern geometric methods for solving complex geometric problems.

Keywords: 3D computer geometric modeling, quadrics, second-order surfaces, nine-point quadric problem, AutoCAD, AutoLisp.

1. Introduction

The geometric solution to the problem of constructing a second-order surface (hereinafter referred to as a quadric) using its nine points has been of interest due to the analogy of this problem with constructing a conic using five points. The construction of a conic based on the Pascal theorem is a well-known problem in geometry courses [1]. A similar quadric problem in its "geometrically accurate" formulation (only using a compass and a ruler) has not been solved, although due attention has been given to it historically [2]. The problem is still being discussed, despite the belief that there is no "geometrically accurate" solution to the problem [3, 4, 5].

Since the 19th century, there have been algorithms [4] in which going beyond the restrictions of "geometrical accuracy" has allowed the authors to propose a solution to the quadric problem. [4] considered several similar algorithms. Our paper deals with one of them

– the Engel algorithm (Engel J. H.) [6]. The content of the algorithm is given in [4]. We will consider the distinct features of the experimental implementation of the algorithm. The algorithm allows us to construct a conic or a section of the desired quadric, using nine points. The center, axes, and surface of the quadric are determined based on this conic.

The use of the algorithm provides for the construction of conics by five points, finding the intersection points of the conics with a straight line and a plane, as well as mutual intersection of the conics. Since such constructions were impossible until recently, the algorithm showed only the fundamental possibility of constructing a quadric.

The practical implementation of the Engel algorithm has become possible in modern CAD using programming tools and 3d computer modeling. In particular, we developed and used such tools when studying quadrics [7–9]. Accumulated experience has allowed us to turn to the experimental implementation of the quadric problem and supplement the works [3–5].

The main problem which arises in the experimental implementation of the Engel algorithm is the appearance of imaginary points during solving when straight lines intersect with conics. It is recommended that imaginary points be substituted with their real analogs [3–5, 10]. However these recommendations should be verified experimentally. Even if such a substitution is possible, it complicates the constructions significantly.

Our preliminary experiments have shown that there are always real solutions to the quadric problem without the formation of imaginary points. Our task is to determine the region and conditions for the implementation of real solutions.

The purpose of the work is as follows: 1 – to implement experimentally the Engel algorithm for constructing a quadric defined by nine points; 2 – to determine and explore the region of real solutions to the quadric problem; 3 – to bring the solution to a visual image of the surface.

2. Methods and software

The experiments were carried out in the AutoCAD suite. The AutoLisp language was used for programming.

The construction of a quadric according to the Engel algorithm provides for multiple repetitions of homogeneous operations. To this end, we developed several special programs. The algorithm cannot be based only on the main capabilities of the AutoCAD suite and AutoLisp programs. The specifications of our special programs are shown below.

Program 1. The construction of a conic by five points [11] implements the "exit to space" method [12, 13]. Using Pascal's hexagon, we find tangents at two out of five points (points *p1, p2*) according to the algorithm [1, p. 168]. We construct a circular cone taking the tangents for contour straight line generators. We select one of the remaining points, for example, point *p3*, and erect the perpendicular to the plane of five points until it intersects with the cone at point *p3**. We construct a section of the cone by the plane *p1, p2, p3** and then orthogonally project the section onto the plane of the defined five points. The projection is the desired conic. As compared to the original version [11], the program is fitted with algorithms for the optimal selection of the points *p1, p2, p3* regulating the length of the resulting conic branches. The conic type also determines its metric parameters – axes, center, focuses, asymptotes of hyperbolas, parabola directrix. The error of constructing a conic by this program is at the level of $10^{-5}...10^{-8}$ [14].

Program 2. The intersection of a conic with a straight-line segment or a plane. We divide the conic branches into 500...1000 segments. We find an intersecting segment, additionally divide it into 20...50 parts, and specify the intersection points until we achieve the required accuracy.

Program 3 determines the intersection points of two conics. We divide each conic branch into 500...1000 segments. We find intersecting pairs of segments. Then we interrupt the cal-

culation if the number of such pairs has reached four and additionally divide the segments of the found pair into 20…50 parts and specify the intersection points.

Program 4 constructs auxiliary conics $c1, c2$ (see Section 4 below). We find their intersection with the straight lines m, n. We construct the conic $c3$ and determine the parameter Δ, on which the problem solution accuracy depends.

Program 5. The construction of fields defining the boundaries of the real or imaginary solution (see Fig. 3, *a*), with color indexing depending on the types of the resulting conics $c1, c2$ and the intersection options of these conics with straight lines m, n.

Program 6. Constructing the series of values of the parameter Δ and determining the parameters kF^*, kG^*, at which $\Delta = 0$, i.e., the accurate solution is achieved.

Program 7 determines the axis of the quadric in the iterative movement cycle of the cutting plane according to the algorithm. This brings it to a position when the perpendicular dropped from the center of the quadric to the plane coincides with the segment connecting the centers of the quadric and the conic of the section.

Programs 2–7 implement iterative construction cycles until the specified and controlled error is achieved. The permissible error was taken at the level of 10^{-3} of the determined value.

In addition to the above programs, we used the library of geometric functions of the AutoLisp suite, for example, the construction of the intersection point of straight lines, as well as the functions of cyclic, algebraic, and logical calculations.

We used the method proposed in [3–5] which compares the obtained experimental objects with predetermined control objects. The control objects include a quadric, a section, a conic, a center point, and quadric axes. One- and two-sheet hyperboloids, a hyperbolic paraboloid, and an ellipsoid were considered as control quadrics. In this paper, we give examples of two control quadrics: a one-sheet hyperboloid; and a hyperbolic paraboloid.

Nine points were defined on the surface of the control quadric. Such point definition automatically takes into account the restrictions [3–6] to their relative position.

The accuracy (or error) of the experiments, which depends on the errors at all construction stages, was estimated by the deviation of the constructed distinguished points of objects from their control values. For example, this could be the distance between the vertices of the experimental and control conics, by the angle between the found and control axis of the quadric, inter alia.

In our paper, the quadric is constructed in the following sequence:

• According to the Engel algorithm, construct a conic for the defined nine points as a section of the desired quadric by the plane defined by points 1, 2, 9;

• Construct two more conics using the first conic and two randomly chosen triples of points from the remaining six defined points. Consider the constructed three conics as a geometric determinant of the desired quadric;

• Find the center of the quadric by the determinant;

• Find the principal axes of the quadric by its determinant and center;

• Using the determinant and axes of the quadric, construct its frame and surface.

3. Control objects and coordinate axis

Let us therefore construct the control objects: a quadric and nine points on its surface. We arbitrarily number (name) the points. We construct the control conic as a section of the control quadric by a plane defined by points 1, 2, 9 (Fig. 1, *a*). We selected these points according to the Engel algorithm. We hide ("freeze") the control objects. There are nine points left. In our experiment, the control quadric is a one-sheet elliptic hyperboloid, the control conic is the ellipse *e*, and the selected points have the coordinates given in Table 1.

Table 1. Coordinates of points for the one-sheet hyperboloid

Point #	X	Y	Z
1	10.700	3.287	13.755
2	−9.056	5.895	28.399
3	15.736	−8.092	36.611
4	5.463	7.694	8.821
5	−2.736	−10.869	2.989
6	−15.380	4.495	7.180
7	-9.672	9.553	35.470
8	−3.667	−11.445	37.209
9	1.692	−6.008	14.478

<u>Statement of Problem.</u> We have nine points which are randomly numbered 1....9. Assuming that the points define a quadric, we should construct a conic as a section of the quadric by a plane defined by points 1, 2, 9.

a)   b)
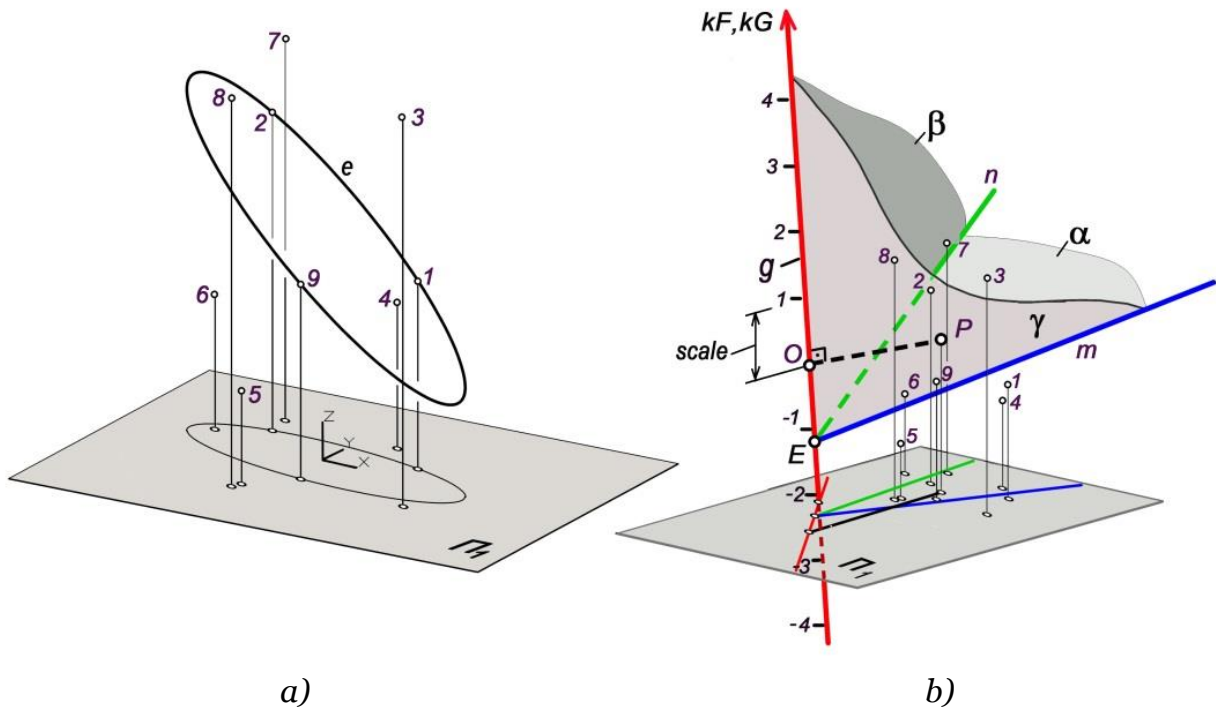
Fig. 1. The defined nine points and control conic (*a*). Initial constructions and the coordinate axis (*b*).

According to the Engel algorithm, we perform the following constructions. Let us consider planes defined by triples of points: α (1,2,9); β (3,4,5); γ (6,7,8). We then construct the edges and the vertex of the trihedral angle formed by these planes. The edges of the angle are straight lines m (α ∩ γ); n (α ∩ β); g (β ∩ γ). The vertex of the angle is point E (Fig. 1, *b*).

The Engel algorithm defines two points F and G on the straight line g. The points are defined by dimensionless coordinates kF and kG. Let us introduce the average parameters for the region of the defined nine points P and *scale*. Point P (P_x, P_y, P_z) is the center point of the region of the nine defined points. For example, $P_x = \frac{\sum_1^9 p_{xi}}{9}$, where p_{xi} are the x-coordinates of the defined nine points. *Scale* is the average size of the region of the defined points. *Scale* = $\frac{\sum_1^9 dist(p_i, \ P)}{9}$, where $dist(p_i, \ P)$ is the distance from the *i*-th point p_i to point P. In the example considered, in the initial ("world") coordinate system (see Fig. 1, *a*), point P (−0.769, −0.610, 20.546). *Scale* = 17.388.

In order to define points *F, G,* let us introduce the axis *kF, kG* on the straight line *g* (Fig. 1, *b*). The origin of coordinates, point O, is defined as the foot of the perpendicular dropped to the straight line *g* from point *P*. The axis is directed opposite to the vertex *E*. The unit of measurement is the *scale* value. Points *F* and *G* will be defined on the straight line *g* by the parameters *kF* and *kG*: $F = scale \times kF$ and $G = scale \times kG$.

4. Auxiliary conics. Real and imaginary solutions

Let us conduct a preliminary experiment. We set the parameters *kF* and *kG* and use them to define points *F* and *G* on the straight line *g* (Fig. 2, *a*). For example, *kF* = 4.453, *kG* = − 4.467. Based on program 1 (see Section 2), we construct the conic *c1* using points 3, 4, 5, F, G and the conic *c2* using points 6, 7, 8, F, G. The conics *c1, c2* have two common points *F, G*. Depending on the position of the points, each conic can be an ellipse or a hyperbola. A parabola can be formed in exceptionally rare cases. In our example, *c1, c2* are two hyperbolas (Fig. 2).
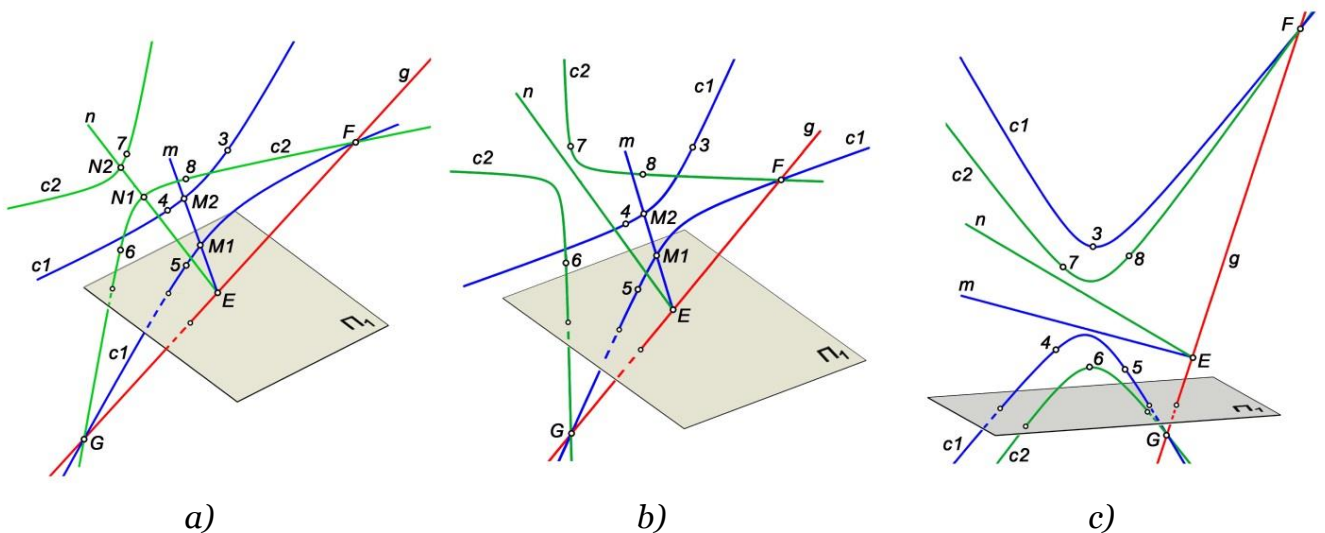

a)   b)   c)

Fig.2. The intersection of the straight lines *m, n* with the conics *c1, c2*: *a* – two real intersections; *b* – a real and an imaginary intersection; *c* – two imaginary intersections.

Using program 2, we find the intersection points *M1, M2* of the straight line *m* with the conic *c1* and the intersection points *N1, N2* of the straight line *n* with the conic *c2*. We thus obtain the following results. All four points are real (Fig. 2, *a*), i.e., the straight lines intersect each of their conics at two points. One or both intersection points may be missing. So-called imaginary points or imaginary intersections are formed. Real and imaginary intersections can appear in various combinations. For example, at the parameters *kF*=3, *kG* =−3, a real intersection was formed with the hyperbola *c1* and the straight line *m* and an imaginary intersection was formed with the hyperbola *c2* and the straight line *n* (Fig. 2, *b*). At *kF*=6, *kG* = −1.5 there are no real intersections, i.e., two imaginary intersections (Fig. 2, *c*) appeared.

There may be no real intersection due to the insufficient length of the hyperbola or parabola branches. Such solutions will be called "conditionally real". In order to identify them, it is sufficient to determine that there is no imaginary intersection.

5. The region of real solutions

A real solution of the quadric problem is possible only for the real intersections of the conics *c1, c2* with the straight lines *n, m*. We determined the region of real solutions using programs 2–5. We put markers on the rectangular region *kF, kG* (Fig. 3, *a*) with a step of 0.25…0.5, which color was determined by the types of the conics *c1, c2* and their intersection option with the straight lines *m, n*.

Table 2. Color of the region markers (to Fig. 3, *a*; Fig. 10, *a*)

Marker	Conic types	Option
(cyan)	E+E, E+H, H+H	1
(orange)	E+H, H+H	2
(blue)	E+E	3
(green)	H+H	3
(red)	E+E	4
(purple)	E+H	4
(magenta)	H+H	4

Conic types: E – ellipse; H - hyperbola. Intersection options:
1 – both real;
2 – a real and a conditionally real;
3 – a real and an imaginary;
4 – two imaginary

If both intersections are real, the marker is blue (option 1), while the conics can be ellipses or hyperbolas in various combinations. If one or both conics are hyperbolas with short branches, and it is sufficient to increase the length of the branches to obtain a real intersection, the marker is orange (option 2). The remaining colors of the markers show other options of imaginary intersections. For example, if it is blue color (option 3) – two ellipses were formed, the intersection is real with one of them and imaginary with the other one.

The experiments show that, depending on the conditions of the problem, the solution regions have the form of stripes, islands, or extensive zones on the coordinate plane *kG, kF*. Since points *F* and *G* jointly determine the conic types, the regions are symmetric relative to the diagonal line *kF=kG*. The coincidence of points *F, G* leads to defining the conics *c1, c2* by four points, which is impossible. Therefore, the field is not defined on the diagonal line *kF=kG*.

6. The hyperbola of real solutions

The presence of real intersections is a necessary but insufficient condition for resolving the problem in general. Let us consider the second necessary condition for resolving the quadric problem.

Let us construct the conic *c3* by defined point 1 and found points N1, N2, M1, M2. We construct a line segment *k* (1,2) and determine point A (*k* ∩ *c3*). Let Δ be the distance between points A and 2. If the length of the segment (1, A) is less than the length (1,2), Δ > 0, and vice versa.

According to the Engel algorithm, the conic *c3* should pass through point 2, i.e., the condition Δ=0 should be satisfied. This does not happen if the parameters *kG, kF* are set arbitrarily, i.e., Δ≠0. The condition Δ=0 is achieved by adjusting the parameters *kG, kF*. Let us consider an example.
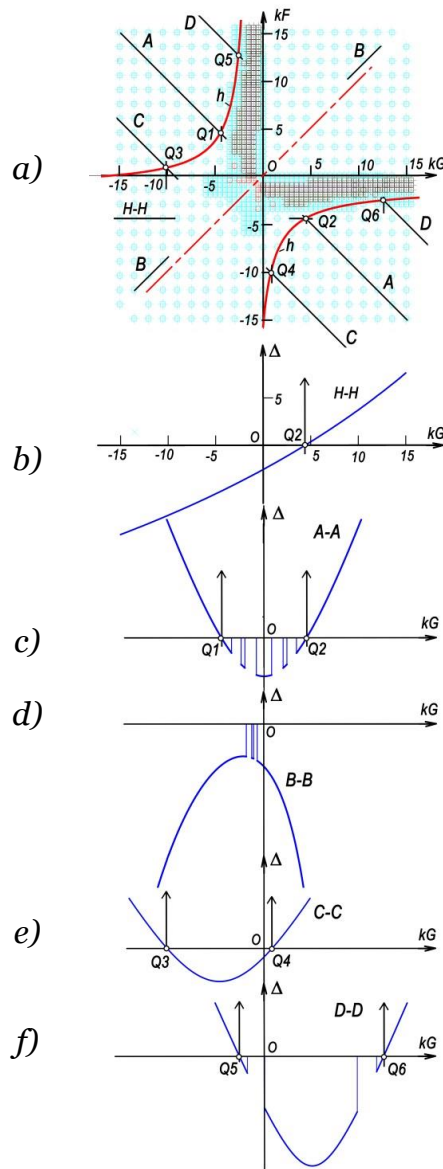
Fig. 3. Field (a) and series (b–f) of real and imaginary solutions.

Let us set the parameters kG, kF in the region of real solutions: kG=9.3, kF=−4.5, construct the conic c_3 and the segment k and find point A. In this example, the conics obtained c_1, c_2, c_3 are hyperbolas. The value of $\Delta(2, A)$ was 3.25 (Fig. 4, a).

Let us shift the point (kG, kF) along the kG axis and set kG= −5, kF=−4.5. The conics c_2, c_3 have become ellipses. Δ =−5.1 (Fig. 4, b).
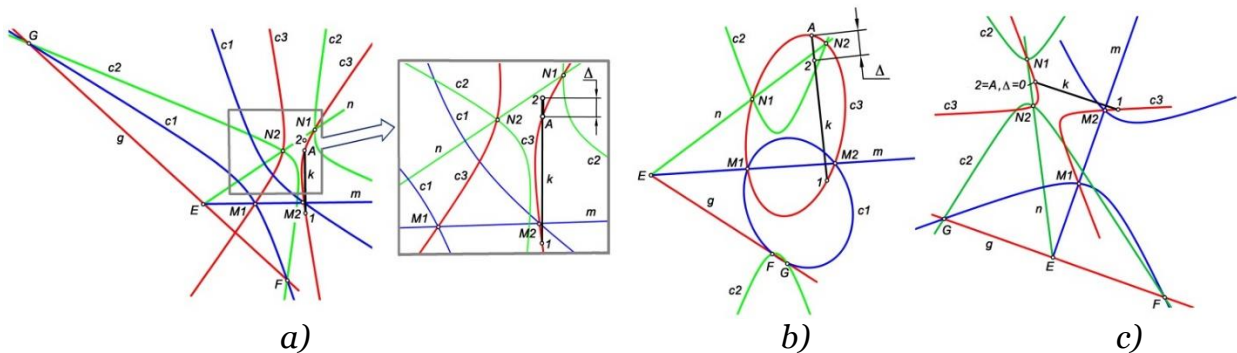

$a)$ $b)$ $c)$

Fig. 4. Determining the parameter Δ: $a - \Delta > 0$; $b - \Delta < 0$; $c - \Delta = 0$.

A change in the sign of Δ in the interval −5 < kG < 9.3 indicates that there are values kG^*, kF^*, at which Δ=0, i.e., the conic $c3$ will pass through point 2. In order to verify this provision and determine kG^*, kF^*, we fixed the value kF and used program 4 to construct a graph of Δ (kG) at $kF = −4.5$. Points kG were set as a horizontal series $H-H$ (Fig. 3, a) passing through the region of real solutions. The resulting dependence Δ(kG) is close to a linear one (Fig. 3, b). kG^* was determined in an iterative cycle. We found the values of $kG^*=4.395$, $kF^*=−4.5$, at which Δ = − 0.000262. The construction of the conics $c1, c2, c3$ at these values led to the practical coincidence of points 2 and A (Fig. 4, c).

The constructions performed at this stage resulted in the conic $c3$ passing through *six* points 1, N1, N2, M1, M2, 2. Points 1, 2 are defined, points $N1, N2$ result from the intersection of the conic $c1$ (3, 4, 5, F*, G*) and the straight line n, points M1, M2 are formed by the intersection of the conic $c2$ (6, 7, 8, F*, G*) with the straight line m. Points F^*, G^* are determined by the conics $1, N1, N2, M1, M2$ passing through point 2.

The set of points kF^*, kG^* obtained at |Δ| < tol define a line of exact real solutions h for the defined set of nine points. Here *tol* is the tolerance for construction errors. In our experiments, *tol* = $1*10^{-3}$.

Numerous experiments with different options of defining 9 points (i.e. with different control quadrics and conics and different numbering of points) have shown that where the adopted coordinate system is used, the line h is approximated by a hyperbola with high accuracy (see also Fig. 10, a). One of the hyperbola axes always coincides with the diagonal straight line $kF=kG$. In the example considered (Fig. 3, a), the imaginary hyperbola axis coincides with the diagonal line.

In order to construct the hyperbola h using program 6, we need to construct four diagonal series of kF (kG) values, determine the dependence Δ (kG) in each of them (Fig. 3), and find points Q(kG^*, kF^*), at which |Δ| < tol. Since the series pass through the region of imaginary solutions, the curves Δ (kG) have gaps. The series A-A, B-B (Fig. 3, c, d) determine the direction of the hyperbola axes and identify the vertices of the hyperbola or points close to them. The series C-C, D-D (Fig. 3, e, f) define four more points Q of the hyperbola h.

7. Final conic and estimate of the construction error

Let us select two points Q on the hyperbola h. The points can be located on the same or different branches of the hyperbola. Each of them should satisfy the condition |Δ| < tol. These can be points used to construct the hyperbola. This condition is satisfied when determining these points. If two new points belonging to the hyperbola are selected, we need to specify their coordinates from the condition |Δ| < tol.

For example, let us select points Q2 ($kG^* = 4.395$, $kF^*= −4.5$) and Q6 ($kG^* = 8.860$, $kF^*= −3.071$) (see Fig. 3, a). We need to repeat the above constructions for each point and obtain the conic $c3$ (1, N1, N2, M1, M2, 2). Let us denote the conic $c3$ for the first point by *c3-1*, and for the second point – by *c3-2* (Fig. 5).
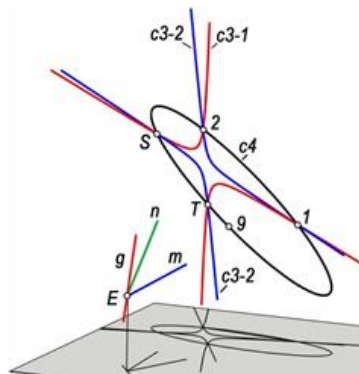

Fig. 5. Final conic *c4*.

Since the points used to construct the conics *c3-1* and *c3-2* belong to the plane α (*m∩n*) = α(1,2,9), the conics intersect at four points. The conics have two common points 1, 2. We need to find two more points of their intersection.

Let us use program 3. After the conics *c3-1, c3-2* are intersected, we find the missing points *S, T* (Fig. 5). If the points *S, T* are closely spaced or missing, we need to choose other points Q on the hyperbola *h* or increase the construction accuracy.

After constructing a conic using points *1, 2, 9, S, T,* we can obtain the final conic *c4* (Fig. 5), which belongs to the surface of the desired quadric. In our example, the conic *c4* is an ellipse.

In order to determine the construction error in our example, we measure the distance between the vertices of the major axis of the constructed *c4* and control *e* (see Fig. 1, *a*) ellipses. The absolute error was 0.0138. At the average size of the region of the defined points *scale* = 17.388, the relative error was ≈0.08%.

If the constructions were carried out without a control conic, then in order to estimate the error, we need to perform constructions for another pair of points Q selected on the hyperbola *h* or change the numbering of six out of the nine defined points keeping plane 1, 2, 9 and repeat the constructions. The error is estimated by comparing the results of two or three constructions.

8. The implementation sequence of the Engel algorithm

Let us summarize the results of the studies and consider the sequence of actions for constructing a conic on the surface of the desired quadric.

1. Prepare the software (see Section 2).

2. In order to test the programs and the algorithm, construct control objects: a quadric, a conic, nine points on its surface, the center and axes of the quadric, which will be used to check the construction accuracy (see Section 3, Fig. 1, *a*).

3. Construct the lines *m, n, g*. Determine the vertex of the trihedral angle *E*, the center of the region of nine points *P*, and the average size of the *scale* region. Taking into account these values, set the coordinate axis on the straight line *g* to specify points *F, G* according to the parameters *kG, kF* (see Section 3, Fig. 1, *b*).

4. Study the field of real solutions (see Sections 4, 5, Fig. 3). This action is advisable but not mandatory.

5. Construct the diagonal series A-A, B-B, etc. (see section 6, Fig. 3). Determine the points of exact solutions Q_1, Q_2 of the hyperbola *h* for each series. It is sufficient to find two points of the hyperbola and their relevant pairs of values (*kF*, kG**), for each of which Δ < 0.001.

6. Construct the conics *c1, c2* for one pair (*kF*, kG**), make sure that Δ < 0.001, and construct the conic *c3-1* (see Section 7).

7. Repeat the constructions for the second pair (*kF*, kG**) and construct the conic *c3-2*.

8. Find points *S, T* and construct the final conic *c4* using points 1, 2, S, T, 9 (see Section 7).

9. Compare the conic *c4* with the control conic or the construction result for another combination of defined points. Based on the comparison results, estimate the solution error (see Section 7).

9. Quadric determinant

Nine points unambiguously define a quadric. However, such a determinant requires complex constructions when solving even the simplest geometric problems. [3−5] proposed the use of a more convenient and visual determinant of three conics, each of which is a section of the desired quadric. Let us consider the creation of such a determinant.
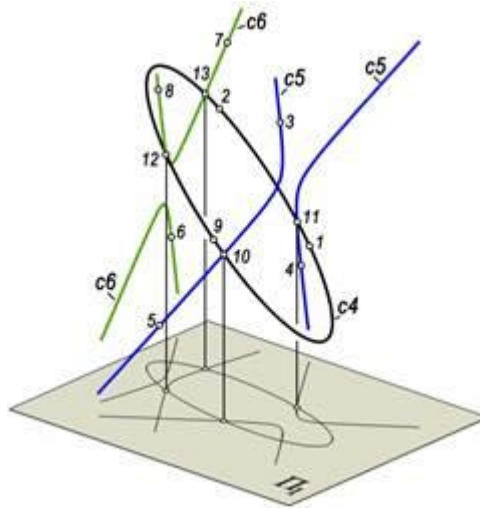
Fig. 6. Quadric determinant.

If the conic $c4$ is constructed on the surface passing through three points 1, 2, 9, we select two triples of points from the remaining six points, for example, 3, 4, 5 and 6, 7, 8. Using program 2, we find the intersection points of the planes defined by these points with the conic $c4$. We obtain two groups of five points: 3, 4, 5, 10, 11 and 6, 7, 8, 12, 13 (Fig. 6). We construct two conics which, jointly with the conic $c4$, form a new determinant of the quadric $c4, c5, c6$. Selecting other triples of points, we obtain numerous variants of the determinant. The intersection of the conics of the determinant by different planes allows us to construct new conics belonging to the surface of the quadric. Such an algorithm is the basis for solving geometric problems with a quadric.

10. Constructing the center of the quadric

There is a known method to find the center of a conic as the intersection point of its two diameters [1, 15, 16]. Let us extend this method to quadrics [5].
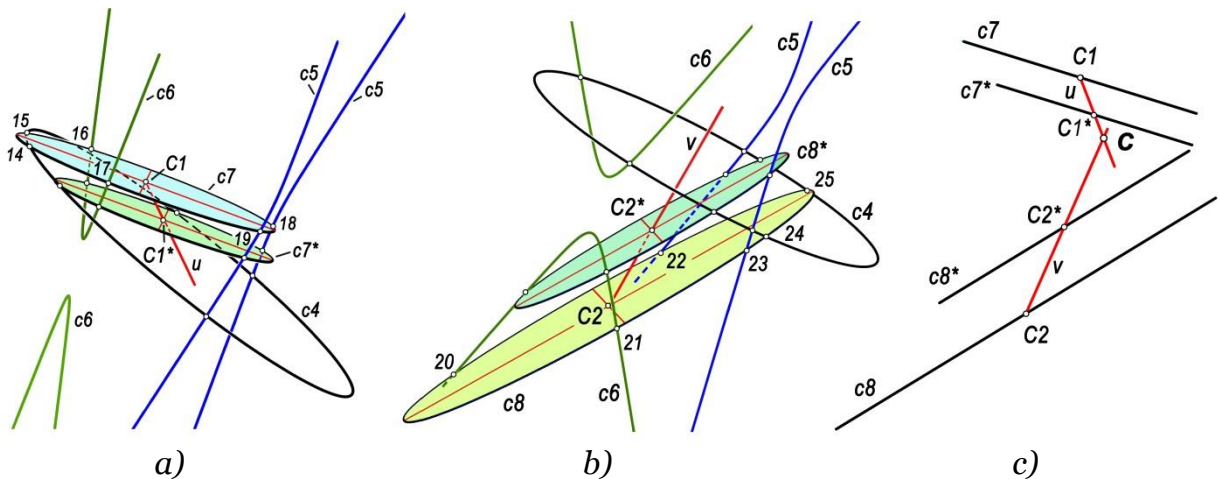

a)   b)   c)

Fig. 7. Constructing the center of the quadric: a, b – constructing two diameters; c – the center as the intersection of diameters.

Let us define the plane $\delta1$ intersecting three conics of the determinant (Fig. 7, a). Using program 2 three times (see Section 2), we find six points 14–19 of the intersection of the plane $\delta1$ with the conics of the determinant. Using arbitrary five out of the found six points, we construct the conic $c7$ and find its center $C1$. We define the second plane $\delta2 \parallel \delta1$ and construct the conic $c7^*$ of the section by the plane $\delta2$ with the center point $C1^*$. Using points $C1$, $C1^*$, we construct a segment of the diameter u.

We define the plane δ_3 intersecting the planes δ_1, δ_2. We find points 20–25 of its intersection with the conics of the determinant and construct the conic c_8 with the center C_2 (Fig. 7, b). We define the next plane $\delta_4 \parallel \delta_3$ and construct the conic c_8^* with the center C_2^* and the segment v (C_2, C_2^*) of the second quadric diameter.

We find point C of the intersection of the quadric diameters defined by the segments u, v, which is the desired center of the quadric (Fig. 7, c).

If the conics of the determinant do not allow us to construct the necessary conics of the sections, we should supplement the determinant with new conics. If the diameters of the quadric u, v are parallel, the center is missing. This indicates that the quadric is one of paraboloids (see Section 12 below).

The diameters u, v are generally formed as skew lines due to construction errors. The length of the segment of the shortest distance between the diameter segments can be used as the error. In the considered example, the length of the error segment was ≈0.060.

If there is a control quadric, the error can be estimated by the average distance between the segments u, v and the center of the control quadric. In the experiment considered, this error was 0.15. The error was 0.80% with respect to the average size of the *scale* region of the defined points.

11. Constructing the principal axes of the quadric

[4,5] proposed a projective algorithm for constructing the principal axes of a quadric. We present our algorithm (program 7) based on the following assumption. If the foot of the perpendicular dropped from the center of the quadric to the section plane coincides with the center of the section conic, the perpendicular coincides with one of the axes of the quadric.
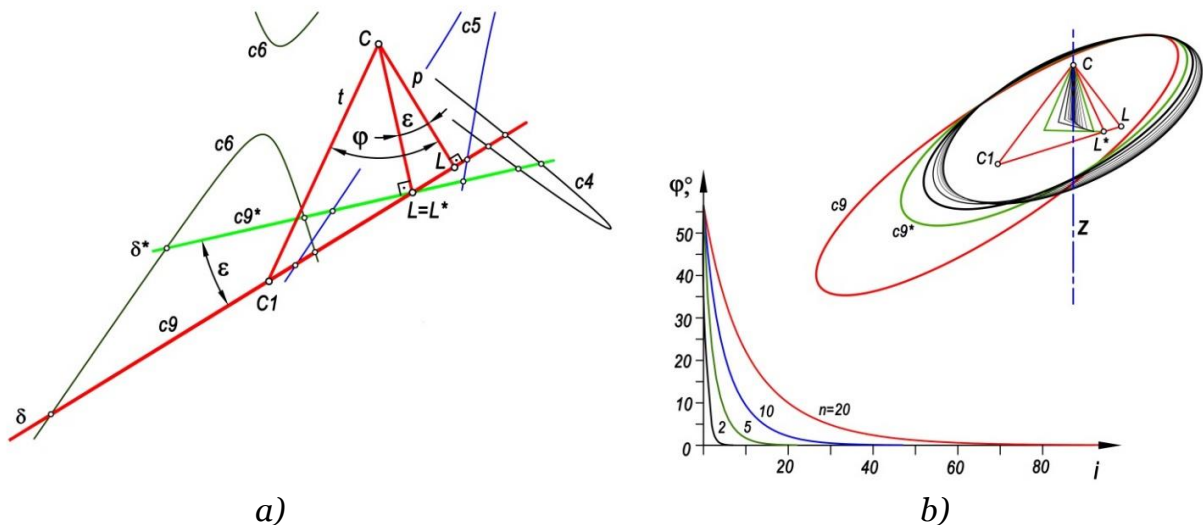

Fig. 8. Constructing the axis of the quadric: a – algorithm; b, c – construction cycle.

<u>Statement of Problem</u> (Fig. 8, *a*). Find the axes of the quadric by the determinant of the quadric, the conics c_4, c_5, c_6, and its center (point C).

1. Define a cutting plane δ and, based on the determinant of the quadric, construct the conic c_9 of the section and determine its center C_1.

2. Let's construct a segment t (C, C1). Drop the perpendicular p from the center of the quadric C to the plane of the conic. The foot of the perpendicular is point L. Measure the angle φ between the segments t, p. Define the angle ε as a fraction of the angle φ, i.e. $\varepsilon = \varphi : n$, where n=2...20.

3. In the plane (t, p), rotate the segment p by an angle ε towards the segment t and find point L^*, to which point L has moved.

4. Define a new plane δ^* so that it passes through point L^* perpendicular to the new position of the segment p. Construct a conic of the section c_9^* by the plane δ^*.

5. Arrange a cycle with iterations i, in which point L is taken as point L^*, then repeat steps 2–4. Control the value of the angle φ in the cycle. It has been established that with an increase in the iterations i, the value of the angle φ continuously decreases (Fig. 8, b). The parameter n determines the smoothness of the angle decrease.

6. Run the cycle (Fig. 8, c) until φ becomes less than the permissible value, for example, 0.01^0. The position of the segments t, p at the end of the cycle is taken as the axis of the quadric z defined with the accepted permissible error.

7. In order to find the other two axes of the quadric by the determinant of the quadric, construct its section perpendicular to the found z-axis. Move the axes of the constructed conic to the center of the quadric. A triple of principal axes of the quadric is formed together with the first axis (Fig. 9, a).

12. Constructing the frame and surface of the quadric

Based on the determinant and the previously found axis of the quadric, we construct its axial and transverse sections. These can be arbitrary sections of the specified direction, in particular, sections by the coordinate planes XZ, YZ, XY. The sections allow us to determine the quadric type (ellipsoid, hyperboloid, etc.). Taking into account the quadric type, we can construct its frame.
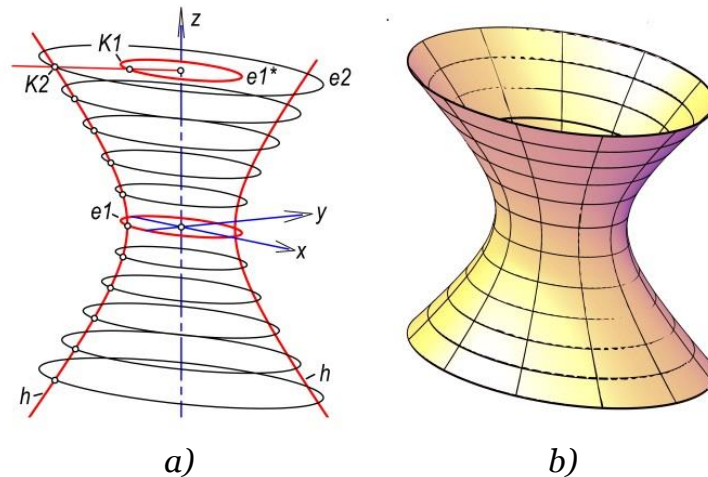

Fig. 9. Frame (a) and surface of the quadric (b).

In the considered example, the axial section is the hyperbola h, the transverse section is the ellipse $e1$, therefore, the quadric is an elliptical one-sheet hyperboloid (Fig. 9, a). Let us construct the frame of the quadric from parallel ellipses. These ellipses are mutually similar [1, 13, 15] and their centers belong to the longitudinal axis of the quadric. In order to construct an arbitrary cross section, for example, the ellipse $e2$, we copy the ellipse $e1$ to the position of the ellipse $e1^*$. In the plane of the ellipse $e1^*$ we draw a straight line from the center of the ellipse to the intersection with the hyperbola h. We scale $e1^*$ to align point $K1$ with point $K2$.

Using the known AutoCAD command, we construct a surface by the frame (Fig. 9, b).

13. Peculiar features of a hyperbolic paraboloid

Let the nine points have the coordinates given in Table 3. Repeating the constructions (see Section 5), we obtain the region of real solutions (Fig. 10, a). The points have the same color marking (see Table 2). The boundaries of the region differ significantly from the previous example (see Fig. 3, a). After constructing several diagonal series, we obtain a line of exact solutions h. Similar to the previous example, this line is approximated by a hyperbola with high accuracy.

Table 3. The coordinates of points for the hyperbolic paraboloid

Point #	X	Y	Z
1	16.762	77.351	110.344
2	89.350	17.000	77.196
3	30.829	0	30.362
4	48.172	17.000	54.467
5	16.400	42.612	68.237
6	73.422	77.351	54.543
7	100.737	37.686	64.341
8	128.805	53.521	39.077
9	50.403	59.529	70.860

Having selected two arbitrary points on the hyperbola h, we implement the Engel algorithm (see Section 8). The constructed conic c_4 is a parabola. The quadric determinant (see Section 8) contains the parabola c_4 and the hyperbolas c_5, c_6 (Fig. 10, b).
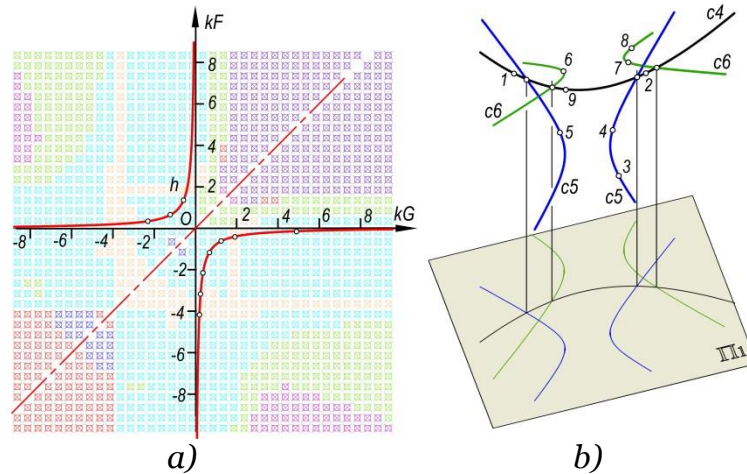

Fig. 10. Hyperbolic paraboloid: a – the region and hyperbola of real solutions; b – the determinant of the quadric.

In order to find the center of the quadric, we constructed two pairs of parallel conics c_7, c_7^* and c_8, c_8^* (Fig. 11, a). The planes of these conics were set parallel to the intersection line of the planes with the conics of the determinant c_5 and c_6. Figure 11, a displays the planes c_5, c_6, c_7, c_7^*, c_8, c_8^* as straight lines. The conics $c_7–c_8^*$ were constructed from the intersection points of the planes of these conics with the quadric determinant. For example, the conic c_7 is constructed using five out of six intersection points 30–35.

We determined the centers of the conics and constructed the diameters of the quadric u (C1, C1*) and v (C2, C2*) using them. We established that, unlike the previous example (see Fig. 7, c), the diameters u, v are mutually parallel, i.e., the center of the quadric, being their intersection point, is missing. This indicates that the quadric is one of the paraboloids.

In order to find the axis of the quadric, we construct a section of the quadric perpendicular to the diameters u, v. In our example, the section is the hyperbola c_9 (Fig. 11, a, b) constructed from points 40–45. The segment ax perpendicular to the plane of the conic c_9 passing through its center C_3 is the Z-axis of the quadric (Fig. 11, b). The axes of the hyperbola c_9 determine the coordinate planes of symmetry ZX and ZY of the desired quadric.
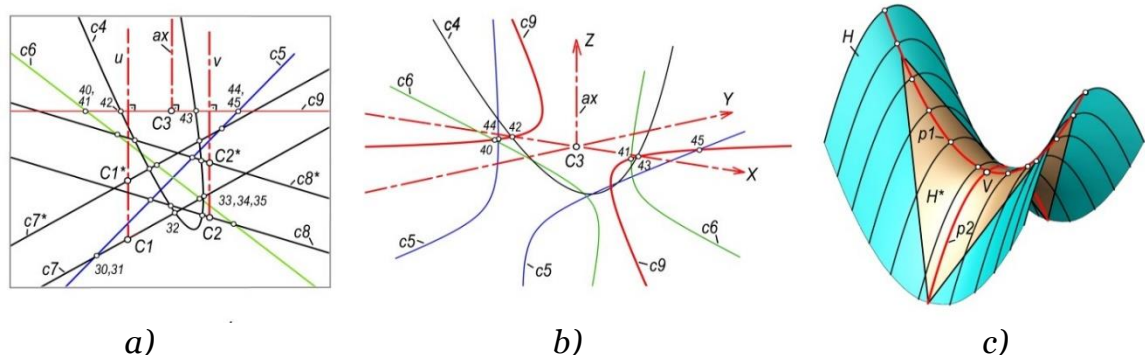
a)   *b)*   *c)*

Fig. 11. Construction of the axes (*a*), vertex (*b*), and surface (*c*) of the hyperbolic paraboloid.

Having constructed the section of the quadric by the coordinate planes, we find the vertex of the quadric V, as well as the parabolas *p1, p2* (Fig. 11, *c*). We take *p1* as the guiding parabola and *p2* – as the generating parabola. We construct the frame of the quadric by the parallel displacement of the parabola *p2* along *p1*. Using the frame, we create the surface of the hyperbolic paraboloid *H*. Cutting out the section *H**, we obtain the well-known form of this surface – a skew plane.

14. Conclusions

1. We developed a program system in the AutoLisp language to construct a quadric defined by nine points using the Engel algorithm. The construction error does not exceed 0.1% of the average sizes of the region of defined points.

2. There is a region of real solutions for any set of nine points (defined subject to the restrictions on their mutual position [3–5]) which does not require operations with imaginary parameters. The region of real solutions contains a line of exact solutions.

3. We developed a method for determining the region and line of exact real solutions. In the adopted coordinate system, the line of exact solutions is approximated by a hyperbola. In order to construct a conic using the Engel algorithm, it is sufficient to select two arbitrary parameter points defined on this line. The arbitrariness of selection shows many possible options for solving the problem leading to a single exact solution.

4. We proposed an interactive algorithm for constructing the principal axes of a quadric.

5. We experimentally confirmed the conclusions 1–4 for the ellipsoid, elliptic and hyperbolic paraboloids, one- and two-sheet hyperboloids.

The lines of our further studies on the construction of a quadric by nine points will cover:

• The theoretical justification of the obtained experimental conclusions about the existence of real solutions in the form of a hyperbola in the Engel algorithm;

• The experimental verification of the potential construction of a quadric, if imaginary intersection points are formed during construction;

• The experimental study and comparative estimate of other well-known algorithms for solving the quadric problem.

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