Scientific Visualization, 2022, volume 14, number 3, pages 107 - 121, DOI: 10.26583/sv.14.3.08

Arithmetic in Functional-Voxel Modeling

Alexey Tolok[1], Natalya Tolok[2]

V.A. Trapeznikov Institute of Control Science of Russian Academy of Sciences,
65 Profsoyuznaya street, Moscow, 117997, Russia

[1] ORCID: 0000-0002-7257-9029, tolok_61@mail.ru
[2] ORCID: 0000-0002-5511-4852, nat_tolok@mail.ru

Abstract

The paper considers the claim that the function-voxel model allows arithmetic operations over the space of values of two different functions given by a single domain. At the same time, there are three possible approaches to solving the problem, leading to a similar result: functional approach - when the analytical representation of functions is involved in the calculations; functional voxel approach - when voxel data representing local geometric characteristics are used in the construction of local functions for further use in the calculation; voxel approach - when exclusively voxel data is used for sequential recalculation of local geometric characteristics of the model. The basic arithmetic operations on functional voxel models are considered, including such procedures as: addition, subtraction, modulo, exponentiation, taking root expressions, multiplication and division of functional voxel models. It is shown that the obtained applicability of arithmetic operations to functional voxel models leads to obtaining new complex functional voxel models.

Keywords: Functional voxel modeling method, arithmetic operations, R-functions.

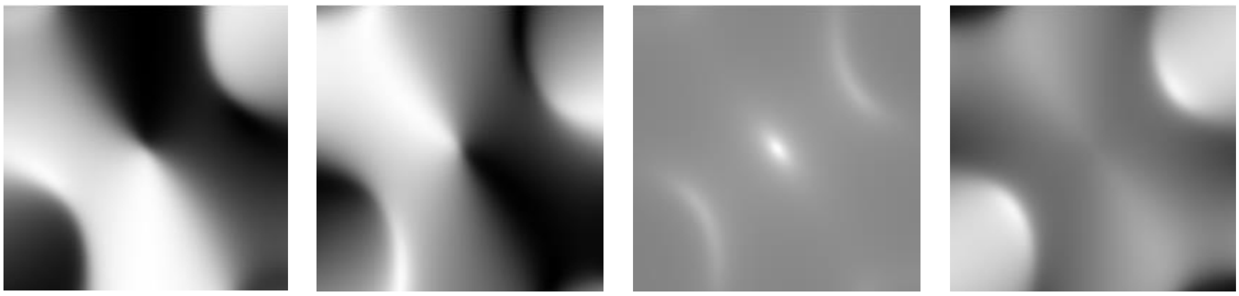
1. Introduction

Various domestic and foreign researchers are engaged in the development of computer representation of the area of the function of the implicit form $F(X_n) = 0$. Such works include studies [1,2], which study the application of R-functional modeling in the construction of grid surfaces for FEM, works [3,4] investigate the method of voxel representation of vector field components for a three-dimensional function in mathematical modeling problems, and many others who worked mainly on the task of rasterization of the function domain for the purpose of zero-boundary allocation and a positive zone of values for use in various problems of geometric layout [5-8].

The method of functional voxel modeling (FVM) was carried out in the system of “RANOK” [9] and is based on the construction of basic graphical M-images that display the characteristics obtained as a result of linear approximation of the surface of the function [10]. By itself, a set of basic images is a graphical display of local geometric characteristics at each point on a given area for some algebraic function. An image that displays some characteristics of an object is further proposed to be called an image-model or an M-image. For example, to obtain a set of two-dimensional base images, the region of the function z = f (x, y) is linearly approximated, where each point in the given region corresponds to the equation of the plane $Ax + By + Cz + D = 0$. Increasing the dimension of the normal to the site, which characterizes the vicinity of the point by one dimension, we obtain the equation $Ax + By + Cz + Dt = 0$. As a result, we have four coefficients (A, B, C, D) for each argument. After normalization by the norm $N = \sqrt{A^2 + B^2 + C^2 + D^2}$ we obtain the components of the normal as local geometric characteristics (n_1, n_2, n_3, n_4) for a local function at each of the points of its domain $z = f(x, y)$:

$$n_1x + n_2y + n_3z + n_4t = 0.$$

The voxel representation of the FV-model assumes mapping each local characteristic into its own image with a monochrome color palette P (for example, in the range [0, 255]). Figure 1 shows basic M-images $(M_1^{IV}, M_2^{IV}, M_3^{IV}, M_4^{IV})$ of local characteristics (n_1, n_2, n_3, n_4), obtained by the formula $M_i = P(1 + n_i)/2, where\ P = 256$. An example was a trigonometric function of the form $z^f = 5(y sin\pi x + x^2 cos\pi y)$, which is displayed by four M-images (Fig. 1). The question arises: is such a computer representation of a function applicable in various calculations and transformations by analogy with an algebraic prototype function?

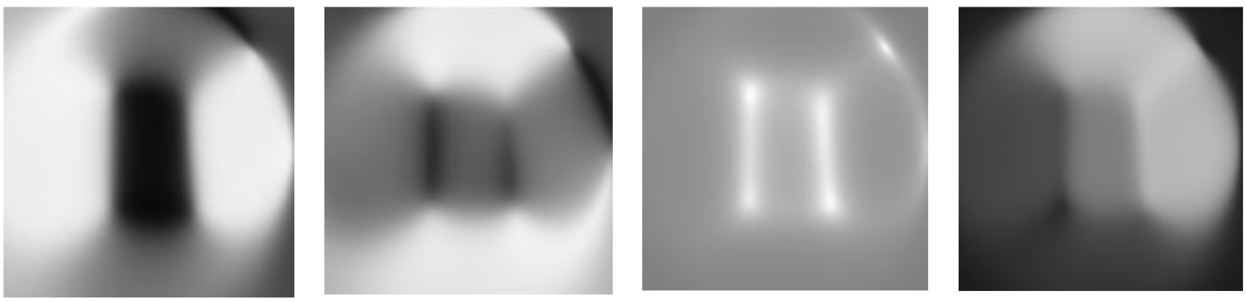


M_1^f M_2^f M_3^f M_4^f

Figure 1: An example of a graphical representation of a functional-voxel model in the area of a trigonometric function f

For further demonstration of arithmetic operations on FV-models, we add to the consideration an exponential function g of the form (Fig. 2):

$$z^g = (x-1)e^{-[x^2+(y+1)^2]} + 10(0{,}2x - x^3 - y^5)e^{-(x^2+y^2)} + e^{-\frac{(x^2+y^2)}{3}}.$$



M_1^g M_2^g M_3^g M_4^g

Figure 2: An example of a graphical representation of a functional-voxel model on the domain of a trigonometric function g

2. Functional-voxel addition and subtraction

For further presentation, let us define the basic constructions of the method, which consist of certain operators. The *FV*-method has the following operators to provide a transition between the analytical and graphical states of the model:

- *Operator G:* $f(x_1, \dots, x_m) = 0 \overset{G}{\rightarrow} g_i(x_{1,i}, \dots, x_{m,i}, n_{1,i}, \dots, n_{m+1,i})$ – operator of approximation of the function $f(x_1, \dots, x_m)$ by the linear function $g_i(x_1, \dots, x_m, n_{1,i}, \dots, n_{m+1,i})$, where i - s the number of voxels in the voxel space of the model. The operator G includes a partition of the function $f(x_1, \dots, x_m)$ on a given domain into finite elements that are the minimum neighborhood of a point in this space. In this case, the number of nodes of the finite element coincides with the dimension of the space. For example, for the function space $x_3 = f(x_1, x_2)$ the final element will be a triangle. In general, the operator G is calculated through the determinant:

$$\begin{vmatrix} x_1 & \cdots & x_m & 1 \\ x_1^1 & \dots & x_m^1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_1^m & \dots & x_m^m & 1 \end{vmatrix} = a_1x_1 + \cdots + a_mx_m + a_{m+1} = 0, ,$$

where x_i^j – i-th coordinate of the j-th node of the finite element.

to reduce the function $n_1x_1 + \cdots + n_mx_m + n_{m+1} = 0$ to the local form, the coefficients a_i are divided by the norm $\sqrt{a_1^2 + \cdots + a_{m+1}^2}$.

- *Operator* $M: (n_1, \dots, n_{m+1}) \xrightarrow{C} (M_1, \dots, M_{m+1})$ – operator of transition from normal field components to graphic representation of voxel space. The general view of the operator can be represented as

$$M_i = \frac{P(1 + n_i)}{2},$$

where P – palette color intensity resolution, and i – function space axis number.

- *Operator* $N: (M_1, \dots, M_{m+1}) \xrightarrow{N} (n_1, \dots, n_{m+1})$ – is the inverse operator of C and is calculated by the formula

$$n_i = \frac{2M_i - P}{P}.$$

- *Operator* $X: (n_1, \dots, n_{m+1}) \xrightarrow{X} x'_i$ – operator defining the value of a function along the selected axis of space

$$x'_i = -\sum_{j=1}^{j=i-1} \frac{n_j}{n_i} x_j - \sum_{j=i+1}^{j=m} \frac{n_j}{n_i} x_j - \frac{n_{m+1}}{n_i},$$

where $x'_i \approx x_i$ the value of the function along the selected i-th axis of space, provided $n_i \neq 0$. It should be noted that checking the condition allows you to assign some minimum value to n_i for example $n_i = 0{,}1 \cdot 10^{-12}$.

There are three possible approaches to solving the problem, leading to a similar result:

- *functional* approach *(F)* - when the analytical representation of the function space is involved in the calculations;
- *functional-voxel* approach *(FV)* - when voxel data characterizing local geometric characteristics are used in the construction of local functions for further participation in the calculation;
- *voxel* approach *(V)* - when only voxel data is used for sequential recalculation of local geometric characteristics of the model.

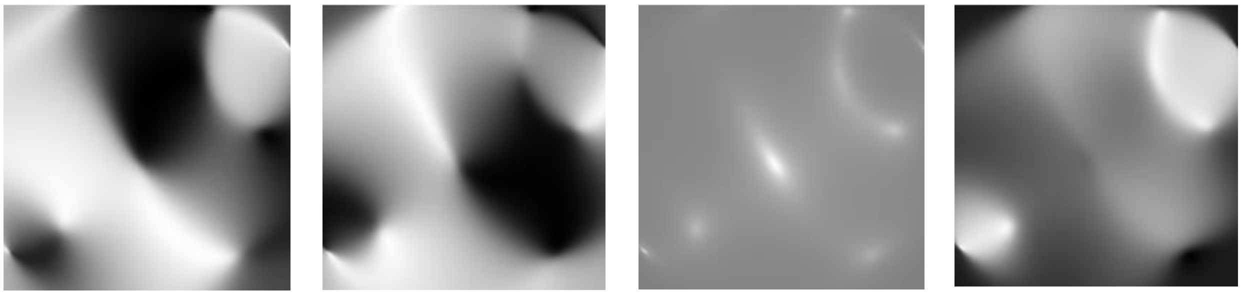
Sum. The space of the sum of two functions with the same specified area can be written as a function $(g + f)(x) = g(x) + f(x), \forall x \in X$. In this case, each of the functions must be presented explicitly: $x_i^f = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$, $x_i^g = g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ then the value of the sum will be written as $x_i^{f+g} = x_i^f + x_i^g$. Consider three approaches to solving the problem using the functional voxel model.

The *functional approach (F)* according to the procedure is comparable to the construction of a new functional-voxel model based on some analytical expression for the sum of functions $f + g$ with the subsequent use of the operators *{G, M}* to construct *FV*-models.

To obtain a graphical representation on a given area for the sum function *(f + g)(x)* using the functional approach (*F*), we write the resulting function as:

$$z^{f+g} = 5(ysin\pi x + x^2cos\pi y) + (x - 1)e^{-[x^2+(y+1)^2]} +$$
$$+10(0{,}2x - x^3 - y^5)e^{-(x^2+y^2)} + e^{-\frac{(x^2+y^2)}{3}}.$$

Using the operators G and M, we obtain the desired graphical representation, which will serve as a template for evaluating the remaining approaches. Figure 3 shows the basic M-images of the sum of values area z^{f+g}.



M_1^{f+g} M_2^{f+g} M_3^{f+g} M_4^{f+g}

Figure 3: Graphic representation on a given area of the sum of two functions, obtained by the functional approach (*F*)

The *functional-voxel approach (FV)* allows you to abandon the calculation of a complex function of the type *(F),* using the voxel data of the *FV*-models (Fig. 1 and Fig. 2) for a given range of functions *f* and *g,* respectively.

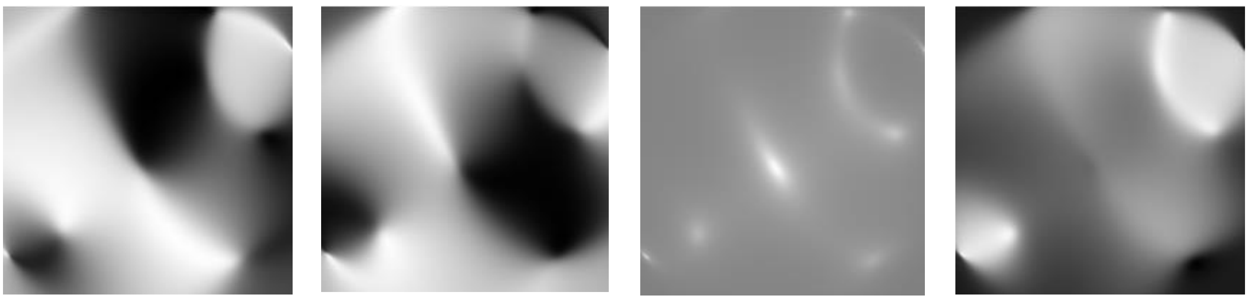
Through the operator

$$N: \left(M_1^f, \dots, M_4^f, M_1^g, \dots, M_4^g\right) \xrightarrow{N} \left(n_1^f, \dots, n_4^f, n_1^g, \dots, n_4^g\right)$$

we obtain a solution in local functions:

$$z^{f+g} = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) + \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right).$$

Further definition of the neighborhood from the obtained points ($z_{i,j}^{f+g}, z_{i+1,j}^{f+g} z_{i,j+1}^{f+g},$) allows us to successively apply the operators *G* and *M* to obtain the desired graphical representation using the functional-voxel approach. Figure 4 shows an example obtained by the FV approach for comparison with the M-images of Figure 3.



M_1^{f+g} M_2^{f+g} M_3^{f+g} M_4^{f+g}

Figure 4: Graphic representation of the space of the sum of two functions, obtained by the *functional-voxel* approach

Before moving on to the third approach, consider a variant that combines two types of function representation in one expression:

$$z^{f+g} = 5(y sin\pi x + x^2 cos\pi y) + \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right).$$

Obviously, the result of the addition to obtain the values of z^{f+g} will be the same as in the cases considered above. This suggests that the proposed computer representation of a function can actively participate in complex analytical calculations on a computer for a given area.

The *voxel* approach *(V)* requires direct voxel transformations, completely refusing to use the operators *G* and *C* in the calculation algorithm. This approach does not require the determination of the neighborhood points and uses only the ratio of the components n_i and, if necessary, the calculation of the value of the local function addend at the point in question on the area.

To describe approach (V), let us consider in more detail the method of obtaining the coefficients of the equation of a plane oriented along three neighboring points:

$$\begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & x_{23} & 1 \\ x_{31} & x_{32} & x_{33} & 1 \end{vmatrix} = a_1x_1 + a_2x_2 + a_3x_3 + a_4 = 0, \text{ where}$$

$$a_1 = x_{12}(x_{23} - x_{33}) - x_{22}(x_{13} - x_{33}) + x_{32}(x_{13} - x_{23}),$$
$$a_2 = -\big(x_{11}(x_{23} - x_{33}) - x_{21}(x_{13} - x_{33}) + x_{31}(x_{13} - x_{23})\big),$$
$$a_3 = x_{11}(x_{22} - x_{32}) - x_{21}(x_{12} - x_{32}) + x_{31}(x_{12} - x_{22}),$$
$$a_4 = -(x_{11}(x_{22}x_{33} - x_{32}x_{23}) - x_{21}(x_{12}x_{33} - x_{32}x_{13}) + x_{31}(x_{12}x_{23} - x_{22}x_{13})).$$

Consider the addition of two neighborhoods at a point expressed by local functions:

$$z^{f+g} = \left(-\frac{a_1^f}{a_3^f}x - \frac{a_2^f}{a_3^f}y - \frac{a_4^f}{a_3^f}\right) + \left(-\frac{a_1^g}{a_3^g}x - \frac{a_2^g}{a_3^g}y - \frac{a_4^g}{a_3^g}\right).$$

Let's rearrange the right side and get

$$z^{f+g} = -x\left(\frac{a_1^f}{a_3^f} + \frac{a_1^g}{a_3^g}\right) - y\left(\frac{a_2^f}{a_3^f} + \frac{a_2^g}{a_3^g}\right) - \left(\frac{a_4^f}{a_3^f} + \frac{a_4^g}{a_3^g}\right).$$

The operator G equally splits the domain into neighborhoods for both functions f and g. The value of coordinates (x_i, y_i) for points for a neighborhood in the plane Oxy coincides for both terms of the functions. Hence, it can be argued that $a_3^f = a_3^g$ (or $a_i^f = a_i^g$ for the general case). Thus, the coefficients a_3^g and a_3^f in the denominator of the expression can be replaced by the general coefficient a_3, which does not depend on the coordinate z, which determines the position of the points of the neighborhood relative to the expressed axis Oz. This means that the expression simplifies form:

$$z^{f+g} = -x\left(\frac{a_1^f + a_1^g}{a_3^f}\right) - y\left(\frac{a_2^f + a_2^g}{a_3^f}\right) - \left(\frac{a_4^f + a_4^g}{a_3^f}\right), \text{from here}$$
$$a_1^{f+g} = a_1^f + a_1^g, \quad a_2^{f+g} = a_2^f + a_2^g,$$
$$a_3^{f+g} = a_3^f = a_3^g \text{ and } a_4^{f+g} = a_4^f + a_4^g.$$

In the normalized state $n_3^f \neq n_3^g$, since the normalization procedure involves all the coefficients of the corresponding function equation in the calculation, and they are different for each function. Therefore, rearranging the normalized expression leads to

$$z^{f+g} = -x\left(\frac{n_1^f}{n_3^f} + \frac{n_1^g}{n_3^g}\right) - y\left(\frac{n_2^f}{n_3^f} + \frac{n_2^g}{n_3^g}\right) - \left(\frac{n_4^f}{n_3^f} + \frac{n_4^g}{n_3^g}\right) =$$
$$= -x\left(\frac{n_1^f n_3^g + n_1^g n_3^f}{n_3^f n_3^g}\right) - y\left(\frac{n_2^f n_3^g + n_2^g n_3^f}{n_3^f n_3^g}\right) - \left(\frac{n_4^f n_3^g + n_4^g n_3^f}{n_3^f n_3^g}\right).$$

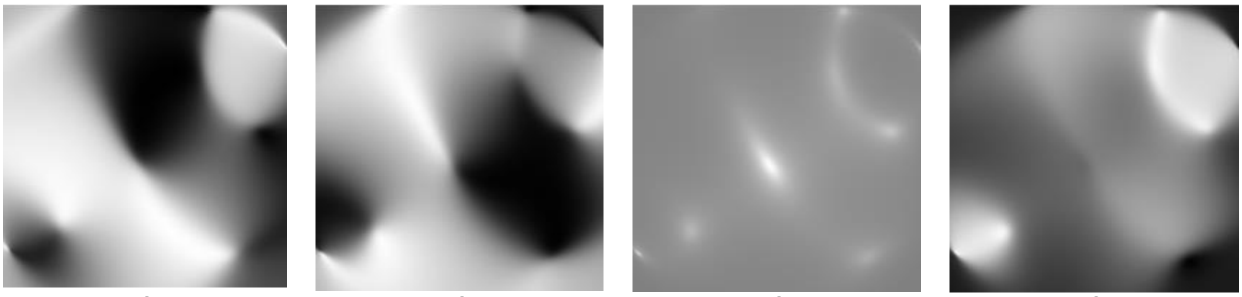
Hence, we can conclude that now

$$a_1^{f+g} = n_1^f n_3^g + n_1^g n_3^f, \quad a_2^{f+g} = n_2^f n_3^g + n_2^g n_3^f,$$
$$a_3^{f+g} = n_3^f n_3^g \quad \text{and} \quad a_4^{f+g} = n_4^f n_3^g + n_4^g n_3^f.$$

This leads to the renormalization of the coefficients

$$n_i^{f+g} = \frac{a_i^{f+g}}{\sqrt{\left(a_1^{f+g}\right)^2 + \left(a_2^{f+g}\right)^2 + \left(a_3^{f+g}\right)^2 + \left(a_4^{f+g}\right)^2}}, \text{where } i - 1 \dots 4.$$

Figure 5 shows the M-images of the graphical representation for the sum of the functions f and g, obtained by the *voxel* approach.



M_1^{f+g} M_2^{f+g} M_3^{f+g} M_4^{f+g}

Figure 5: Graphical representation of the sum of two functions obtained by the *voxel* approach

In the general case, we have

$a_i^{f+g} = n_i^f n_i^g$, $a_j^{f+g} = n_j^f n_i^g + n_j^g n_i^f$, где $j = 1, \dots, i-1, i+1, \dots, m$.

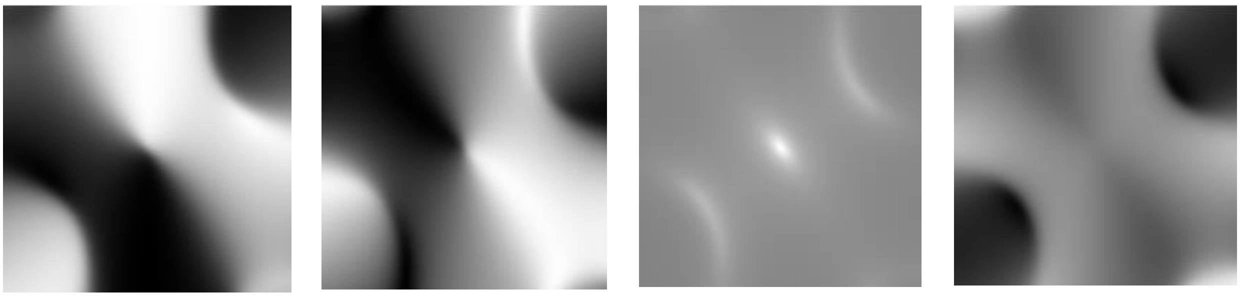
$$n_k^{f+g} = \frac{a_k^{f+g}}{\sqrt{\sum_{l=1}^{m} \left(a_l^{f+g}\right)^2}}, \qquad \text{where} \quad k = 1, \dots, m.$$

Subtraction. The subtraction can be represented as a sum with the negation of a function, which means that it is enough to consider ways to obtain a negation construction $f(x) = -f(x)$. Figure 6 shows the voxel model for the function

$z^{-f} = -\left(5(y sin\pi x + x^2 cos\pi y)\right)$.

Considering the *functional-voxel* representation, we obtain

$$z^{-f} = -\left(-\frac{a_1^f}{a_3^f}x - \frac{a_2^f}{a_3^f}y - \frac{a_4^f}{a_3^f}\right) = -\frac{-a_1^f}{a_3^f}x - \frac{-a_2^f}{a_3^f}y - \frac{-a_4^f}{a_3^f}.$$



M_1^{-f} M_2^{-f} M_3^{-f} M_4^{-f}

Figure 6: Graphical representation of the negation of the function $f(x) = -f(x)$, obtained by the *functional* approach

Figure 6 clearly shows that the M-images C_1^{-f}, C_2^{-f} and C_4^{-f} in relation to the images C_1^f, C_2^f and C_4^f have an inverted (inverse) color $C_i^{-f} = P - C_i^f$ (where P is the maximum value of the palette). *The M-image* C_3^{-f} in relation to the M-image C_3^f retains its color values, which is confirmed by the expression obtained above.

3. Functional-voxel exponentiation

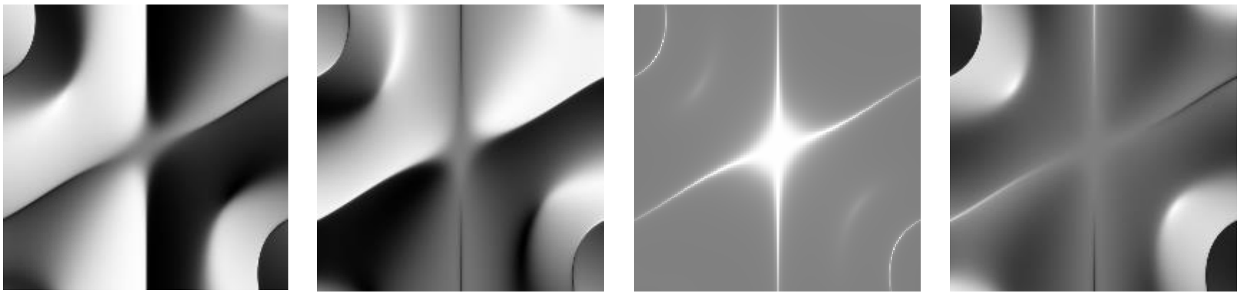
Exponentiation. The operation of raising to a power can be considered on the procedure of squaring with further generalization.

The square power of f can be written as $f^2 = f \cdot f$,

or

$$f^2 = \left(5(y sin\pi x + x^2 cos\pi y)\right)^2.$$

Figure 8 shows the *FV*-model for calculating such a function using the functional approach.



$M_1^{f^2}$ $M_2^{f^2}$ $M_3^{f^2}$ $M_4^{f^2}$

Figure 8: Graphical representation of a function f^2

We will proceed from the FV-representation of the function f^2 by local functions:

$$f^2 = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right)\cdot\left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right).$$

Let's simplify the expression

$$f^2 = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right)\cdot z^f = -\frac{n_1^f z^f}{n_3^f}x - \frac{n_2^f z^f}{n_3^f}y - \frac{n_4^f z^f}{n_3^f}.$$

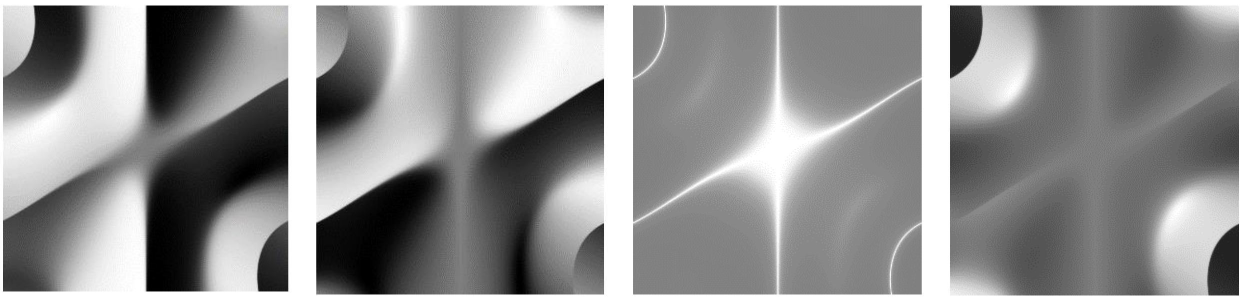
from here

$$a_1^{f^2} = n_1^f z^f,$$
$$a_2^{f^2} = n_2^f z^f,$$
$$a_3^{f^2} = n_3^f,$$
$$a_4^{f^2} = n_4^f z^f.$$

This leads to the renormalization of the coefficients

$$n_i^{f^2} = \frac{a_i^{f^2}}{\sqrt{\left(a_1^{f^2}\right)^2 + \left(a_2^{f^2}\right)^2 + \left(a_3^{f^2}\right)^2 + \left(a_4^{f^2}\right)^2}}, \text{where } i - 1 \ldots 4.$$

Figure 9 shows images of a graphical representation for a given area of the function *f*, squared by the *voxel* approach.



$M_1^{f^2}$ $M_2^{f^2}$ $M_3^{f^2}$ $M_4^{f^2}$

Figure 9: Graphical representation of a function in a square, obtained by a *voxel* approach

Consider the reverse action - root extraction for the resulting FV-model $\sqrt{f^2}$. For this, we write down that

$$\sqrt{f^2} = \frac{f^2}{f} = \frac{\left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right)^2}{\left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right)},$$

or

$$\sqrt{f^2} = \frac{\left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right)^2}{\sqrt{z^{f^2}}}.$$

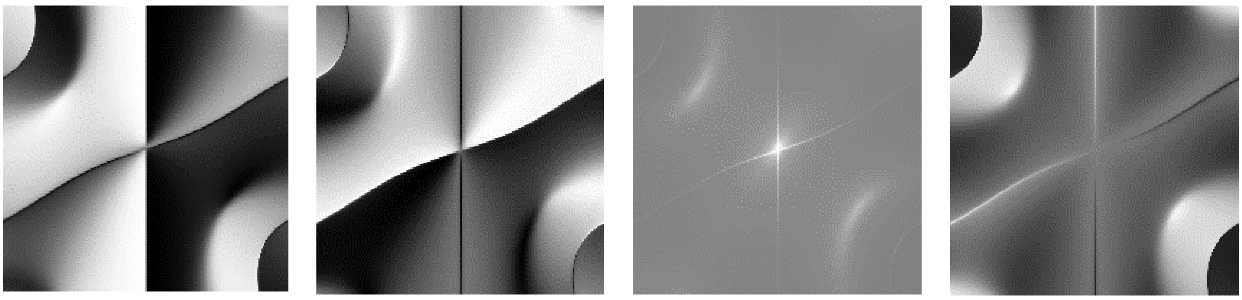
From here

$$a_1^{\sqrt{f^2}} = n_1^{f^2},$$
$$a_2^{\sqrt{f^2}} = n_2^{f^2},$$
$$a_3^{\sqrt{f^2}} = n_3^{f^2}\sqrt{z^{f^2}},$$
$$a_4^{\sqrt{f^2}} = n_4^{f^2}.$$

This leads to the renormalization of the coefficients

$$n_i^{\sqrt{f^2}} = \frac{a_i^{\sqrt{f^2}}}{\sqrt{\left(a_1^{\sqrt{f^2}}\right)^2 + \left(a_2^{\sqrt{f^2}}\right)^2 + \left(a_3^{\sqrt{f^2}}\right)^2 + \left(a_4^{\sqrt{f^2}}\right)^2}}, \text{where } i - 1 \dots 4.$$

Figure 10 shows the M-images of the graphical representation of a given area for the function $\sqrt{f^2}$, obtained by the voxel approach.

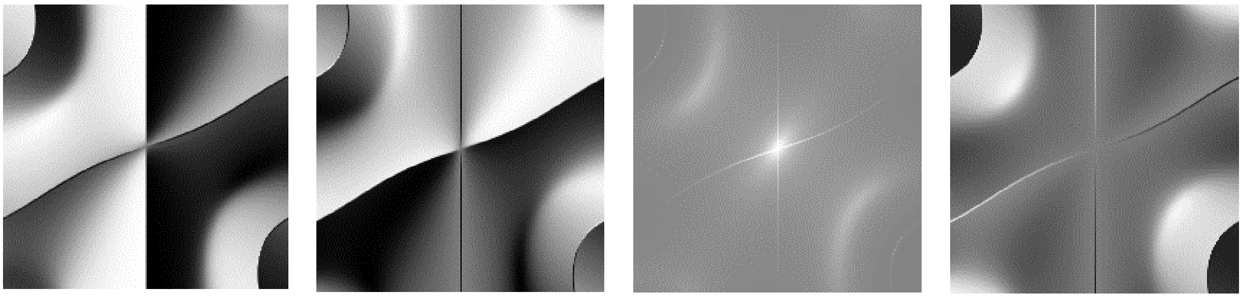


$M_1^{\sqrt{f^2}}$ $M_2^{\sqrt{f^2}}$ $M_3^{\sqrt{f^2}}$ $M_4^{\sqrt{f^2}}$

Figure 10: Graphical representation of the result of extracting the square root of the function f^2, obtained by the *voxel* approach

For comparison, Figure 11 shows an example of a FV-model obtained from the calculation of the function

$$z^{\sqrt{f^2}} = \sqrt{\left(5(y sin\pi x + x^2 cos\pi y)\right)^2}.$$



$M_1^{\sqrt{f^2}}$ $M_2^{\sqrt{f^2}}$ $M_3^{\sqrt{f^2}}$ $M_4^{\sqrt{f^2}}$

Figure 11: Graphical representation of the extraction of the square root of the function f^2, obtained by the *functional* approach

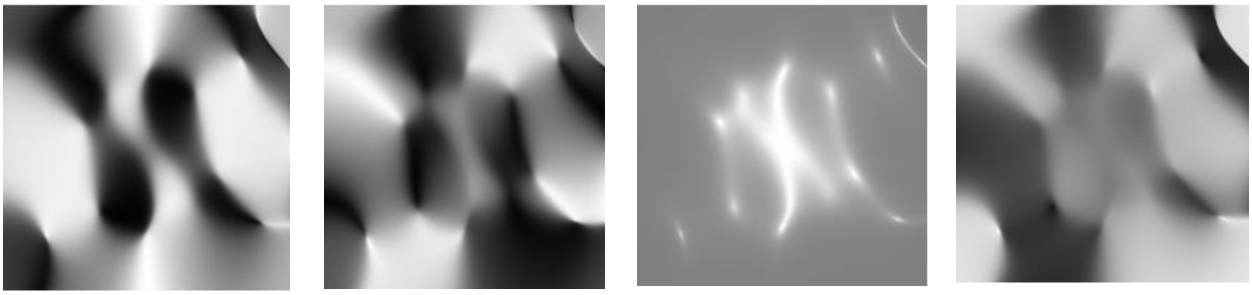
The proposed result can also refer to the result of *taking modulo*.

4. Functional-Voxel multiply and divide

Multiplication. Similarly, consider the arithmetic procedure for multiplying functions by three possible approaches. As an example, let's keep the same functions: f - trigonometric and g - exponential. Then the multiplication function for the first approach takes the form:

$$z^{fg} = \left(5(ysin\pi x + x^2 cos\pi y)\right) \cdot \left((x-1)e^{-[x^2+(y+1)^2]} + 10(0{,}2x - x^3 - y^5)e^{-(x^2+y^2)} + e^{-\frac{(x^2+y^2)}{3}}\right).$$

Using the operators G and M, we obtain the required graphical representation (Fig. 12).



M_1^{fg} M_2^{fg} M_3^{fg} M_4^{fg}

Figure 12: Graphic representation of the product of two functions, obtained by the *functional* approach

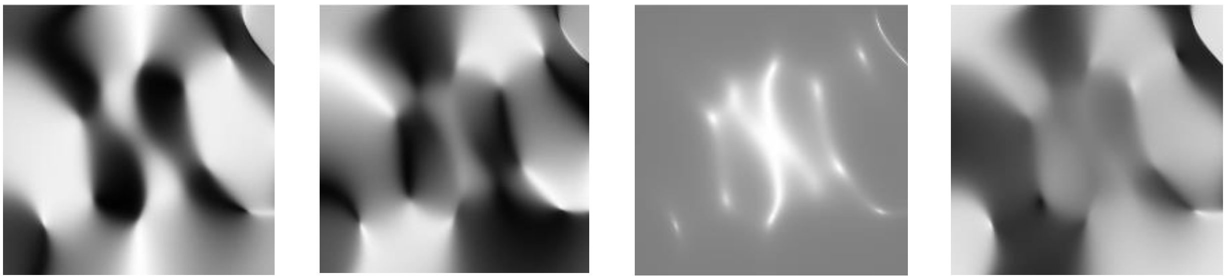
The functional-voxel approach (FV), as noted earlier, allows you to abandon the calculation of a complex function of the previous type, using voxel data of the graphical representation of the functions f and g, respectively. Through the operator

$$N: \left(M_1^f, \dots, M_4^f, M_1^g, \dots, M_4^g\right) \xrightarrow{N} \left(n_1^f, \dots, n_4^f, n_1^g, \dots, n_4^g\right)$$

we obtain a solution in a normal equation of the form for the operator X:

$$z^{fg} = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) \cdot \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right).$$

Further definition of the neighborhood of the point z^{fg} allows us to successively apply the operators G and M to obtain the desired graphical representation by the *functional-voxel approach*. Figure 13 shows a set of M-images obtained by the *FV-approach* for the domain of the product of values.



M_1^{fg} M_2^{fg} M_3^{fg} M_4^{fg}

Figure 13: Graphic representation of the multiplication of functions obtained by the *functional-voxel* approach

The *voxel approach (V)* to multiplication leads to the determination of normalized coefficients based on the equation

$$z^{fg} = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) \cdot \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right).$$

Let's change the calculation strategy and write that

$$z^{fg} = \left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) \cdot z^g$$

and

$$z^{fg} = z^f \cdot \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right).$$

Both cases should lead to a single solution, which means

$$\left(-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}\right) \cdot z^g + z^f \cdot \left(-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}\right) = 2x_3^{fg}.$$

Let's open the brackets and rearrange the resulting expression:

$$-\frac{n_1^f}{n_3^f}x \cdot z^g - \frac{n_2^f}{n_3^f}y \cdot z^g - \frac{n_4^f}{n_3^f} \cdot z^g - \frac{n_1^g}{n_3^g}x \cdot z^f - \frac{n_2^g}{n_3^g}y \cdot z^f - \frac{n_4^g}{n_3^g} \cdot z^f =$$

$$= -\frac{n_1^f}{n_3^f}x \cdot z^g - \frac{n_1^g}{n_3^g}x \cdot z^f - \frac{n_2^f}{n_3^f}y \cdot z^g - \frac{n_2^g}{n_3^g}y \cdot z^f - \frac{n_4^f}{n_3^f} \cdot z^g - \frac{n_4^g}{n_3^g} \cdot z^f =$$

$$= -x\left(\frac{n_1^f}{n_3^f} \cdot z^g + \frac{n_1^g}{n_3^g} \cdot z^f\right) - x_2\left(\frac{n_2^f}{n_3^f} \cdot z^g + \frac{n_2^g}{n_3^g} \cdot z^f\right) - \left(\frac{n_4^f}{n_3^f} \cdot z^g + \frac{n_4^g}{n_3^g} \cdot z^f\right) =$$

$$= -x\left(\frac{n_1^f n_3^g z^g + n_1^g n_3^f z^f}{n_3^f n_3^g}\right) - y\left(\frac{n_2^f n_3^g z^g + n_2^g n_3^f z^f}{n_3^f n_3^g}\right) -$$

$$-\left(\frac{n_4^f n_3^g z^g + n_4^g n_3^f z^f}{n_3^f n_3^g}\right) = 2z^{fg}$$

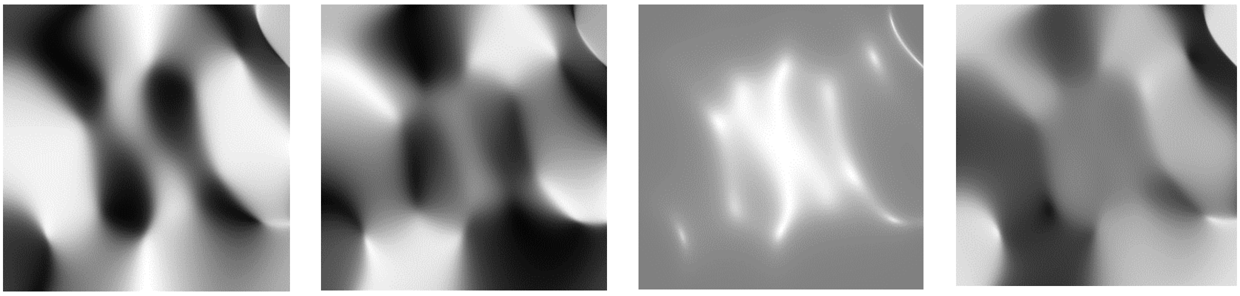
Hence, we can conclude that

$$a_1^{fg} = n_1^f n_3^g z^g + n_1^g n_3^f z^f,$$
$$a_2^{fg} = n_2^f n_3^g z^g + n_2^g n_3^f z^f,$$
$$a_3^{fg} = 2n_3^f n_3^g,$$
$$a_4^{fg} = n_4^f n_3^g z^g + n_4^g n_3^f z^f.$$

Obtaining the components leads to the renormalization of the coefficients

$$n_i^{fg} = \frac{a_i^{fg}}{\sqrt{\left(a_1^{fg}\right)^2 + \left(a_2^{fg}\right)^2 + \left(a_3^{fg}\right)^2 + \left(a_4^{fg}\right)^2}}, \text{where } i - 1 \dots 4.$$

Figure 14 shows the M-images of the graphical representation for the product of the functions f and g, obtained by the *voxel approach*, where you can see a complete match with Figure 13.



M_1^{fg} M_2^{fg} M_3^{fg} M_4^{fg}

Figure 14: Graphic representation on a given area of the product of two functions, obtained by the *voxel* approach.

In the general case, we have

$$a_i^{fg} = n_i^f n_k^g x_k^g + n_i^g n_k^f x_k^f, (i = 1 \dots k-1, k+1 \dots m),$$

$$a_k^{fg} = n_k^f n_k^g,$$

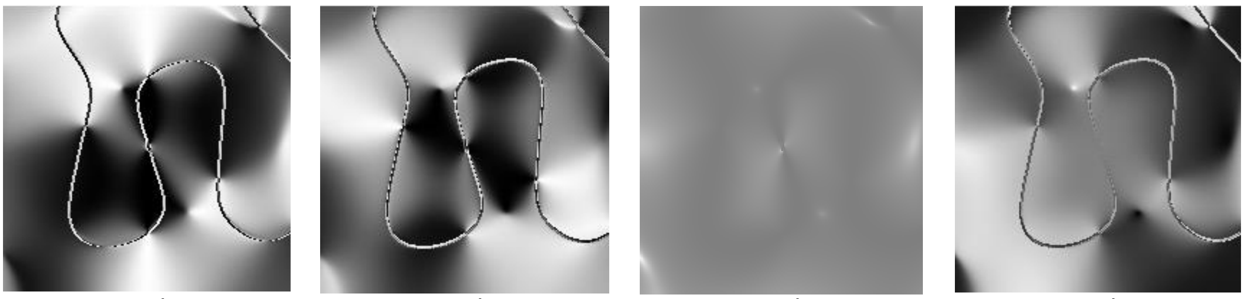
$$n_i^{fg} = \frac{a_i^{fg}}{\sqrt{\sum_{l=1}^{m} \left(a_l^{fg}\right)^2}}, \quad \text{где} \quad i = 1, \dots, m.$$

Divide. Consider the arithmetic procedure for dividing functions in three possible approaches. The functional dividing for the first approach will take the form:

$$z^{div} == \frac{5(y sin\pi x + x^2 cos\pi y)}{(x-1)e^{-[x^2+(y+1)^2]} + 10(0{,}2x - x^3 - y^5)e^{-(x^2+y^2)} + e^{-\frac{(x^2+y^2)}{3}}},$$

$$g \neq 0.$$

Using the operators *G* and *M*, we obtain desired graphical representation shown in Figure 15.



M_1^{div} M_2^{div} M_3^{div} M_4^{div}

Figure 15: Graphical representation on a given area of the quotient of two functions, obtained by the *functional* approach

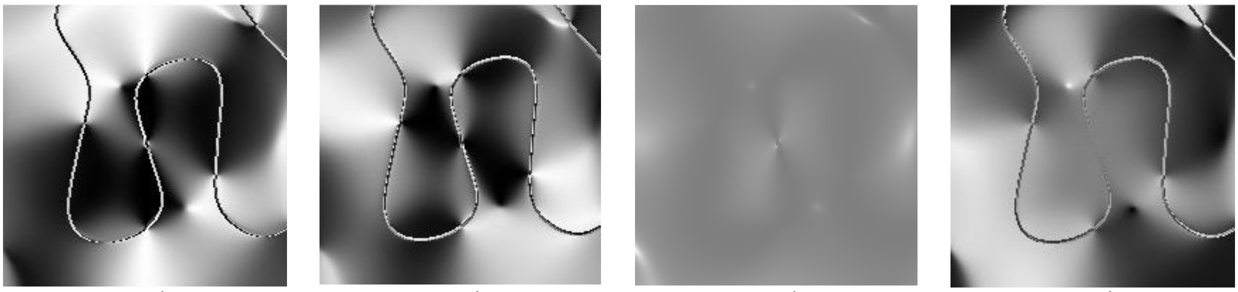
Functional voxel approach (FV). Through the operator

$$N: \left(M_1^f, \dots, M_4^f, M_1^g, \dots, M_4^g\right) \xrightarrow{N} \left(n_1^f, \dots, n_4^f, n_1^g, \dots, n_4^g\right)$$

we obtain a solution in a normal equation of the form for the operator X :

$$z^{div} = \frac{-\frac{n_1^f}{n_3^f}x - \frac{n_2^f}{n_3^f}y - \frac{n_4^f}{n_3^f}}{-\frac{n_1^g}{n_3^g}x - \frac{n_2^g}{n_3^g}y - \frac{n_4^g}{n_3^g}},\ g \neq 0.$$

Further definition of the neighborhood of the point $x_3^{f/g}$ allows us to successively apply the operators G and M to obtain the desired graphical representation by the *FV-approach*. Figure 16 shows the M-images obtained by the *FV-approach,* where you can see a complete match with Figure 15.



$M_1^{f/g}$ $M_2^{f/g}$ $M_3^{f/g}$ $M_4^{f/g}$

Figure 16: Graphical representation on a given area of the quotient of two functions, obtained by the *functional-voxel* approach

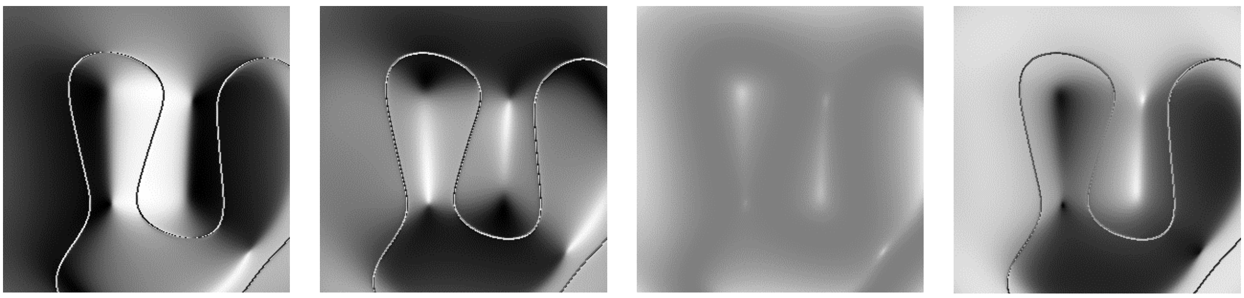
The *voxel* approach *(V)* of functional dividing leads to the determination of normalized coefficients based on the equation just considered.

We proceed from the fact that the quotient can be represented by the product:

$$z^{div} = \frac{f}{g} = f \cdot \frac{1}{g}.$$

Having previously obtained the *FV*-model for the $1/g$, function, you can use the product procedure that was just discussed above.

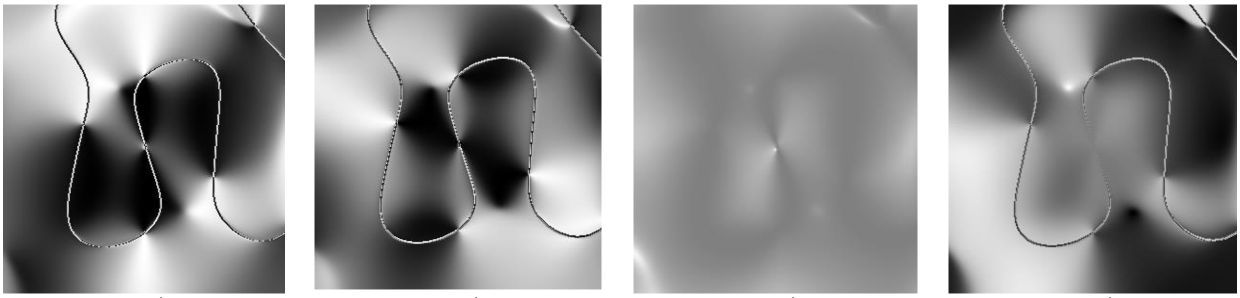
Figure 17 shows the M-images of the graphical representation for the private function $1/g$, obtained by the functional approach (*F*).



$M_1^{1/g}$ $M_2^{1/g}$ $M_3^{1/g}$ $M_4^{1/g}$

Figure 17: Graphical representation on a given area for the $1/g$ function, obtained by the *functional* approach

Figure 18 shows the M-images obtained as a result of the voxel approach to the product $f\left(\frac{1}{g}\right)$. A visual assessment of the result obtained with the results in Figures 16 and 17 allows us to speak about the correctness of the formulas obtained for calculating the *V- divide*.



$M_1^{f/g}$ $M_2^{f/g}$ $M_3^{f/g}$ $M_4^{f/g}$

Figure 18: Graphical representation on a given area for the f/g function, obtained by the *voxel* approach

5. Conclusion

The proposed set of approaches for the implementation of means of constructing and calculating analytically presented implicit functions on a given domain allows us to create a new computer computing platform that makes it possible, based on simulated local geometric characteristics, to significantly simplify the representation of complex functions domain to its local analogues (local functions) on a computer and, based on them, to organize complex computational constructions in the form of functional expressions. An example is the FV-modeling of R-functions that allow to obtain set-theoretic operations on the domain of two analytical expression [11, 12], etc. Also, the implementation of basic arithmetic procedures allows us to talk about the possibility of using voxel models in solving functional equations, avoiding complex formulations.

References

[1] Lysniak A.A, Choporov S.V., Gomeniuk S.I. Metodika vizualizacyi geometricheskih obyektov, opisannyh s pomoshyu R-funkcyi // Vestnik Zaporozhskogo nacionalynogo universyteta : Sbornik nauchnyh statey. Fiziko-matematicheskie nauki. – 2010. – №1. – S. 88-97.

[2] Choporov S.V., Lysniak A.A,, Gomeniuk S.I. Ispolzovanie funkcyi V.L. Rvacheva dlya geometricheskogo modelirovaniya oblastey slozhnoy formy // Prikladnaya informatika. – 2010. – №2 (26). – S. 109-122.

[3] Myltsev O.M., Kondratyeva N.O., Leontyeva V.V. Funkcionalnaya model osnovnyh biznes-processov systemy “RANOK” // Vestnik Zaporozhskogo nacionalynogo universyteta : Sbornik nauchnyh statey. Fiziko-matematicheskie nauki. – 2018. – №2. – S. 88-99.

[4] Myltsev O., Pozhuyev A., Leontieva V., Kondratieva N. The Assessment of the Complexity of the Recursive Approach to Voxelization of Functionally Defined Objects in the Euclidean Space En. *International Journal Of Mathematics And Computer Research*. 2020. Volume 8. Issue 03. P. 2028-2034. (Index Copernicus).

[5] E Vin Tun, Markin L.V., Turbin N.Yu. Geometricheskiy podhod k ocenke ergonomichnosti proektiruemyh izdeliy // Izvestya TulGU. Tehnicheskie nauki. – 2021. – Vyp. 4. – S. 469-478.

[6] Markin L.V., Korn G.V., Kui Min Han, [16] E Vin Tun Diskretnye modely geometrycheskogo modelirovanya komponovki aviacionnoy tehniki // Trudy MAI. – 2021. – Vyp. 86. – S. 1-30.

[7] Akenine-Moller, T. Real-Time Rendering. Third Edition / T. Akenine-Moller, E. Haines, N. Hoffman. – Wellesley: A. K. Peters, Ltd., 2008. – 1027 p.

[8] Crassin, C. GigaVoxels: Ray-Guided Streaming for Efficient and Detailed Voxel Rendering / C. Crassin, F. Neyret, S. Lefebvre, E. Eisemann // ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games (I3D), feb 2009.

[9] Maksimenko-Sheyko K.V., Sheyko T.I., Tolok A.V. R-funkcii v analiticheskom proektirovanii s primeneniem sistemy “RANOK” / Vestnik MGTU Stankin. M.: MGTU “Stankin”, 2010. № 4. S. 139-151.

[10] Tolok A.V. Lokalnaya komputernaya geometria : uchebnoe posobie / A.V. Tolok. – Moskva : Ay Pi Ar Medya, 2022. – 152 s.

[11] Tolok A.V., Tolok N.B., Sycheva A.A. Construction of the Functional Voxel Model for a Spline Curve / Proceedings of the 30th International Conference on Computer Graphics and Machine Vision (GraphiCon 2020, St.Petersburg). Sankt-Peterburg: CEUR Workshop Proceeding, 2020. Vol-2744. C. http://ceur-ws.org/Vol-2744/paper52.pdf.

[12]Maksimenko-Sheyko K.V. R-funkcii v matematicheskom modelirovanii geometricheskih objektov I fizicheskih poley : monographia / K.V. Maksimenko-Sheyko. – Harkov : Izd-vo IPMash NAN Ukrainy, 2009. – 306 s.