Scientific Visualization, 2022, volume 14, number 1, pages 1 - 17, DOI: 10.26583/sv.14.1.01

Irregular Curves in Engineering Geometry and Computer Graphics

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Abstract

Graphically-defined irregular curves are found in various engineering problems. To use such a curve in the design process, it is replaced (approximated) by an analytical function. The article considers traditional approach when a graphically-defined curve is approximated by cubic Bezier segments (with unit weight coefficients) connected to each other by the order of smoothness G2 (with a continuous change in curvature). It is shown that for planes, the well-known algebraic condition of a G2-smooth connection of Bezier segments reduces to the solution of an ordinary quadratic equation. An algorithm is obtained that can be used to control the shape of a planar composite Bezier curve without violating the specified order of smoothness. The algorithm differs in that it allows for variation of both directions of tangents at the junction points and the radii of curvature at the end points of the composite curve. In particular, the algorithm can be used to find the equation of a planar cubic Bezier segment defined by tangents and radii of curvature at their end points. The calculation of the coordinates of the control points of such a segment is reduced to solving a system of two quadratic equations or constructing the intersection points of two parabolas.

The problem of G2-smooth conjugation of two straight lines, a straight line and a circle, and two circles (with predetermined conjugation points) is considered. An example of construction of a G2-smooth closed contour touching given straight lines and having a given curvature at the closing point is presented. An experiment on the approximation of a physical spline of a composite cubic Bezier curve is performed. The approximation error was less than 2%.

Keywords: composite cubic Bezier curve, Bernstein polynomial, physical spline, curvature, approximation, smoothness, degree of freedom.

1. Introduction

Graphically defined irregular curves are found in various engineering problems. For example, an undulating curve drawn arbitrarily by an architect can become the basis of a project (Fig. 1). For the practical application of such a curve, it is replaced (with a certain degree of accuracy) by a regular curve.

Fig. 1. Modern airport

In modern CAD systems, the irregular curve is approximated by the NURBS curve, which has become a standard tool for computer modeling [1]. This approach may not always satisfy the designer. The NURBS curve, despite its universality, does not take into account certain local geometric conditions imposed on the simulated line (tangents at the nodal points, radii of curvature at the end points, etc.). For example, using Fusion 360 CAD tools, it is impossible to construct a curve which smoothly mates two given circles at the indicated mating. Meanwhile, for a set of parametrized cubic curves, this problem can have four solutions (see paragraph 5).

Another approach is based on the use of a composite curve passing through the characteristic points of the simulated line and satisfying the specified smoothness conditions. Bezier [2, 3] or Hermite [4] cubic curves are most often used as segments of a composite curve. The choice of cubic curves is explained by the simplicity and clarity of their mathematical description combined with good "flexibility", sufficient for many practical applications. In particular, when using cubic Bezier segments with unit weight coefficients, the error in modeling the physical spline does not exceed 2% (see paragraph 6).

The main problem in the formation of a composite curve is to ensure a given degree of smoothness. Let us assume that the simulated irregular curve has a degree of smoothness G^2 (continuous change in curvature). The riverbed, the trajectory of the aircraft, a flexible metal ruler, Euler elastics [5], and other natural curves have a smoothness of at least G^2. Even the movement of a pencil on paper, as a body of non-zero mass, obeys Newton's second law, according to which a jump in the acceleration vector is possible only with an abrupt change in the driving force. In this case, they say that "the architect's hand trembled".

Segments can be smoothly joined in various ways. In [3], to ensure a given degree of smoothness, the authors proposed to change the order of the connected segments, which leads to complication of the mathematical model. In [4], a simplified approach is used, when at the junction point the vectors of the first derivatives are assumed to coincide not only in direction, but also in modulus. At the same time, one degree of freedom is lost.

The classical approach proposed by P. Bezier is based on connecting segments of the same order [2, p. 169]. In a monograph by A Foks [6, p. 152], a vector condition for a G^2-smooth connection of segments of a spatial cubic Bezier curve is obtained. We show that for the plane case this condition is reduced to the solution of an ordinary quadratic equation (see paragraph 3).

The problem of forming a planar cubic Bezier segment with predetermined tangents and radii of curvature at both ends is also worth investigating and solving. In [6, p. 153] it is noted that in order to construct such a segment, it is necessary to solve an algebraic equation of the fourth degree. In this article we show that there is no need to solve the equation of the fourth degree, since the task is reduced to finding the intersection points of two parabolas (see paragraph 5).

Scientific novelty. A graphoanalytic algorithm has been developed for the construction of a flat composite G^2-smooth (everywhere twice differentiable) cubic Bezier curve passing through specified reference points and touching the specified straight lines at these points. A distinctive feature of the algorithm is to take into account the direction of tangent vectors and radii of curvature at the reference and end points of the curve being constructed. A software module has been developed to allow for interactive control of the shape of a composite curve (while maintaining the second order of smoothness at the junction points).

The work is beneficial to the scientific community since the theory and practice of forming composite parametrized curves used in technical design since the mid-1960s is not sufficiently reflected in Russian scientific publications and textbooks on engineering and computer graphics [7, 8, 9].

Practical significance. The graphoanalytic algorithm proposed in the article allows for the construction of a G^2-smooth composite cubic Bezier curve with specified tangents at the reference points, and specified tangents and specified radii of curvature at the end points. Such curves are used to model a variety of geometric objects and physical processes, in particular, to approximate irregular (graphically defined) curved lines.

2. Problem statement

An irregular curve passing through the reference points 0, 1, 2,..., n is fixed on the plane. Tangents τ_0, τ_1, τ_2,..., τ_n are marked at the reference points. At the end points 0, n, the curvature K_0, K_n is given. We need to construct a G^2-smooth approximating function which passes through these points, touches these lines, and has a particular curvature at the end points. The permissible approximation error is determined by the constructor; as a rule, it should not exceed 1...2%.

We shall compose the desired curve from segments of cubic curves in the Bezier shape. Any segment with the number i=1, 2, 3, ..., n is completely defined by its characteristic polyline: the starting point i-1, the ending point I, and the control points Q_i, P_i. For example, the first segment 0-1 is described by parametric Bezier equations:

$$\begin{aligned} x(t) &= (1-t)^3 x_0 + 3t(1-t)^2 x_{Q1} + 3t^2(1-t)x_{P1} + t^3 x_1 \\ y(t) &= (1-t)^3 y_0 + 3t(1-t)^2 y_{Q1} + 3t^2(1-t)y_{P1} + t^3 y_1 \end{aligned}, \tag{1}$$

where $Q_1(x_{Q1}, y_{Q1})$, $P_1(x_{P1}, y_{P1})$ are the control points, and points $0(x_0, y_0)$, $1(x_1, y_1)$ are the boundaries of the Bezier segment. The control points Q_1, P_1 are incident to the tangents τ_0, τ_1. The parameter t varies in the range $t \in [0, 1]$. The auxiliary Cartesian coordinate system xy on the drawing plane can be specified arbitrarily. The position of the control points is determined from the condition of continuity of curvature changes at the interface points of neighboring segments, as well as from the condition of providing a predetermined curvature at the boundary points 0, n.

Solution. Let us fix the shape of a segment. Fixation is provided by specifying the characteristic polyline. The number of the fixed segment and its characteristic polyline is set by the constructor. Find the left and right segments smoothly connected to a fixed segment. On the left and right, attach new segments to the resulting segments, each time providing a condition of continuity of curvature at the points of contact. At the same time, we must repeatedly solve two local problems.

Local Problem 1. *Find the control points of a planar cubic Bezier segment 1-2 smoothly connected to a fixed cubic Bezier segment 0-1.* Segments 0-1 and 1-2 are coplanar. At the junction point 1, both segments must have a common tangent τ_1 and a common radius of curvature. In addition, the segment 1-2 to be constructed must touch at its endpoint 2 a predetermined straight line τ_2.

The main difficulty is ensuring the continuity of curvature at the junction of segments. In case of parametrized cubic curves, the solution is reduced to finding the roots of a quadratic equation (see paragraph 3).

We show that the local problem 1 has ∞^1 solutions. The parametric equations of the cubic Bezier segment contain eight scalar coefficients. Therefore, the segment has eight degrees of freedom. The coordinates of the segment boundary points are fixed. The constructed segment has 4 degrees of freedom. The control points must be incident to the specified tangent. This requirement takes away two more degrees of freedom from the desired segment. The two remaining degrees of freedom allow the constructor to specify the radii of curvature at the start and end points of the segment. But in the condition of problem 1, the curvature is fixed only at the butt point 1, which absorbs only one degree of freedom. Consequently, the segment being constructed retains one degree of freedom. Thus, there are ∞^1 Bezier segments satisfying the condition of problem 1, which allows for the shape of the segment to be controlled.

Local Problem 2. *Find the control points of a planar cubic Bezier segment with given tangents and radii of curvature at its end points*. The problem can have 0, 2, or 4 solutions (see paragraph 5). If the signs of curvature are fixed at the end points, then the problem cannot have more than one solution.

3. The condition of continuity of curvature at the interface point of Bezier segments

Construction of a G^2-smooth composite curve passing through these points and touching the given straight lines at these points begins with the fixation of a segment. We shall assume that the user has fixed the first segment 0-1 by presenting it as a parameterized curve $\mathbf{r}^{(1)}(t)$. Here and further, the superscript in parentheses denotes the segment number. The second segment 1-2 must be attached to the first segment 0-1, ensuring that the curvature of the connected segments is equal at the butt point 1 and contact with the given straight lines τ_1, τ_2 at points 1, 2.

The curvature K of a parametrically given curve $\mathbf{r}(t)$ is determined by the expression

$$\mathbf{B}K = \frac{\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|^3}, \qquad (2)$$

where $\mathbf{B}$ is the unit vector of the binormal. If a plane curve is considered, then vector $\mathbf{B}$ is located perpendicular to the plane of the drawing.

It follows from (2) that the condition of equality of curvature of segments $\mathbf{r}^{(1)}(t)$ and $\mathbf{r}^{(2)}(t)$ at the junction point 1 has the form

$$\frac{\dot{\mathbf{r}}_1^{(1)}\times\ddot{\mathbf{r}}_1^{(1)}}{|\dot{\mathbf{r}}_1^{(1)}|^3} = \frac{\dot{\mathbf{r}}_1^{(2)}\times\ddot{\mathbf{r}}_1^{(2)}}{|\dot{\mathbf{r}}_1^{(2)}|^3}. \qquad (3)$$

We shall write the first derivatives as

$$\dot{\mathbf{r}}_1^{(1)} = w_1^{(1)}\mathbf{T}_1,\ \dot{\mathbf{r}}_1^{(2)} = w_1^{(2)}\mathbf{T}_1\ , \qquad (4)$$

where $\mathbf{T}_1$ is the general tangent vector, and the magnitudes $w_1^{(1)}, w_1^{(2)}$ are the modules of the first derivatives at the junction point 1 (the subscript hereafter denotes the point number). Substituting expressions (4) into (3), we obtain the smoothness condition:

$$\mathbf{T}_1\times\ddot{\mathbf{r}}_1^{(2)} = \mathbf{T}_1\times\ddot{\mathbf{r}}_1^{(1)}\left(\frac{w_1^{(2)}}{w_1^{(1)}}\right)^2 \qquad (5)$$

Condition (5) will be met if

$$\ddot{\mathbf{r}}_1^{(2)} = \ddot{\mathbf{r}}_1^{(1)}\left(\frac{w_1^{(2)}}{w_1^{(1)}}\right)^2.$$

But condition (5) will also be fulfilled if any vector collinear to vector $\mathbf{T}_1$ is added to the vector $\ddot{\mathbf{r}}_1^{(2)}$, for example, vector $\mu_1\dot{\mathbf{r}}_1^{(1)}$ where μ_1 is an arbitrary scalar [6, p. 150]. We obtain the smoothness condition

$$\ddot{\mathbf{r}}_1^{(2)} = \ddot{\mathbf{r}}_1^{(1)}\left(\frac{w_1^{(2)}}{w_1^{(1)}}\right)^2 + \mu_1\dot{\mathbf{r}}_1^{(1)}, \qquad (6)$$

valid for any value of μ_1 (both positive and negative). The variable parameter μ_1 gives an additional degree of freedom in the system of constructing a G^2-smooth curve.

3.1. The smoothness condition in the Bezier form

By writing the equation of segment 0-1 in the Ferguson form

$$\mathbf{r}^{(1)}(t) = \mathbf{r}_0(1-3t^2+2t^3)+\mathbf{r}_1(3t^2-2t^3)+\dot{\mathbf{r}}_0 t(1-t)^2+\dot{\mathbf{r}}_1^{(1)}(-t^2+t^3)$$

and differentiating twice, we obtain an expression for calculating the second derivative at the end of the first segment (at t=1):

$$\ddot{\mathbf{r}}_1^{(1)} = 6\mathbf{r}_0 - 6\mathbf{r}_1 + 2w_0\mathbf{T}_0 + 4w_1^{(1)}\mathbf{T}_1, \tag{7}$$

where $w_0\mathbf{T}_0 = \dot{\mathbf{r}}_0,\ w_1^{(1)}\mathbf{T}_1 = \dot{\mathbf{r}}_1^{(1)}$ are derivatives of the first segment at its boundary points 0, 1.

Similarly, by writing the equation of the second segment 1-2 as

$$\mathbf{r}^{(2)}(t) = \mathbf{r}_1(1-3t^2+2t^3) + \mathbf{r}_2(3t^2-2t^3) + \dot{\mathbf{r}}_1^{(2)}t(1-t)^2 + \dot{\mathbf{r}}_2(-t^2+t^3)$$

and differentiating twice, we obtain an expression for calculating the second derivative at the beginning of the second segment (at t=0):

$$\ddot{\mathbf{r}}_1^{(2)} = -6\mathbf{r}_1 + 6\mathbf{r}_2 - 4w_1^{(2)}\mathbf{T}_1 - 2w_2\mathbf{T}_2, \tag{8}$$

where $w_1^{(2)}\mathbf{T}_1 = \dot{\mathbf{r}}_1^{(2)},\ w_2\mathbf{T}_2 = \dot{\mathbf{r}}_2$ are vector derivatives of the second segment at boundary points 1, 2.

Substituting expressions (7), (8) into (6), we obtain a condition for a smooth connection of segments 0-1 and 1-2:

$$3\mathbf{r}_2 + 3\mathbf{r}_1(\lambda_1^2 - 1) - 3\mathbf{r}_0\lambda_1^2 = w_0\mathbf{T}_0\lambda_1^2 + (2w_1^{(2)} + 2\lambda_1^2 w_1^{(1)} + 0.5\mu_1 w_1^{(1)})\mathbf{T}_1 + w_2\mathbf{T}_2, \tag{9}$$

where the designation $\lambda_1 = w_1^{(2)} / w_1^{(1)}$ (the ratio of the modules of vector derivatives 1 $\dot{\mathbf{r}}_1^{(1)}, \dot{\mathbf{r}}_1^{(2)}$ at jointing point) is introduced.

For a smooth connection of segments, it is not necessary to ensure the continuity of changes in the modules of derivatives $\dot{\mathbf{r}}_1^{(1)}, \dot{\mathbf{r}}_1^{(2)}$ [6, p. 165]. The parameter λ_1 can take any positive values.

To write the smoothness condition (9) in the Bezier form, we take into account that the control points Q_1, P_1 of segment 0-1 and the vector derivatives $w_0\mathbf{T}_0 = \dot{\mathbf{r}}_0,\ w_1^{(1)}\mathbf{T}_1 = \dot{\mathbf{r}}_1^{(1)}$ at the end points of this segment are connected by Bezier relations (Fig. 2):

$$\mathbf{q}_1 = \mathbf{r}_0 + \frac{1}{3}w_0\mathbf{T}_0, \quad \mathbf{p}_1 = \mathbf{r}_1 - \frac{1}{3}w_1^{(1)}\mathbf{T}_1, \tag{10}$$

where $\mathbf{q}_1$, $\mathbf{p}_1$ are vectors indicating the position of control points Q_1, P_1. Similarly, the control points Q_2, P_2 of the second segment and the derivatives $w_1^{(2)}\mathbf{T}_1 = \dot{\mathbf{r}}_1^{(2)},\ w_2\mathbf{T}_2 = \dot{\mathbf{r}}_2$ at the end points of this segment are related by the relations:

$$\mathbf{q}_2 = \mathbf{r}_1 + \frac{1}{3}w_1^{(2)}\mathbf{T}_1, \quad \mathbf{p}_2 = \mathbf{r}_2 - \frac{1}{3}w_2\mathbf{T}_2, \tag{11}$$

where $\mathbf{q}_2$, $\mathbf{p}_2$ are vectors indicating the position of the control points Q_2, P_2.

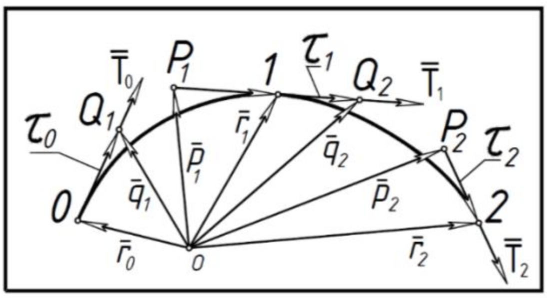

Fig. 2. Connecting Bezier segments

From (10) and (11) it follows:

$$w_0\mathbf{T}_0 = 3(\mathbf{q}_1 - \mathbf{r}_0),\ \ w_1^{(1)}\mathbf{T}_1 = 3(\mathbf{r}_1 - \mathbf{p}_1),\ \ w_1^{(2)}\mathbf{T}_1 = 3(\mathbf{q}_2 - \mathbf{r}_1),\ \ w_2\mathbf{T}_2 = 3(\mathbf{r}_2 - \mathbf{p}_2). \tag{12}$$

Substituting (12) into (9), we obtain a smoothness condition in which there is no vector $\mathbf{r}_2$ indicating the position of the endpoint of the second segment, since the terms involving this vector are reduced to:

$$\mathbf{p}_2 = 2\mathbf{q}_2 + \mathbf{q}_1\lambda_1^2 - \mathbf{p}_1(2\lambda_1^2 + 0.5\mu_1) + \mathbf{r}_1(\lambda_1^2 + 0.5\mu_1 - 1). \quad (13)$$

According to (12), the control vectors $\mathbf{p}_1$ and $\mathbf{q}_2$ are interrelated:

$$\mathbf{q}_2 = (1+\lambda_1)\mathbf{r}_1 - \lambda_1\mathbf{p}_1 . \quad (14)$$

Substituting (14) into (13) (excluding vector $\mathbf{q}_2$), we obtain a condition for a smooth connection of segments 0-1 and 1-2 in the Bezier form:

$$\mathbf{p}_2 = [(1+\lambda_1)^2 + 0.5\mu_1]\mathbf{r}_1 + \mathbf{q}_1\lambda_1^2 - \mathbf{p}_1[2\lambda_1(\lambda_1+1) + 0.5\mu_1]. \quad (15)$$

Here $\mathbf{r}_1$ is a vector indicating the position of the abutting point 1, $\mathbf{q}_1$ and $\mathbf{p}_1$ are vectors indicating the position of the control points Q_1, P_1 of the first (fixed by the user) segment 0-1, $\mathbf{p}_2$ is a vector indicating the position of the control point P_2 of the second (constructed) segment 1-2 (see Fig. 2). In condition (15) it is not required to specify either the position of the starting point 0 of the first segment or the position of the end point 2 of the second segment.

Thus, if the vector $\mathbf{p}_2$ indicating the position of the control point P_2 of the second segment satisfies condition (15), then, regardless of the position of points 0 and 2, segments 0-1 and 1-2 will have the same curvature at the junction point (with fixed vectors $\mathbf{r}_1$, $\mathbf{q}_1$, $\mathbf{p}_1$ and arbitrarily specified values of parameters λ_1 and μ_1). The position of the control point Q_2 (vector $\mathbf{q}_2$), according to the relation (14), functionally depends only on λ_1 (with fixed vectors $\mathbf{r}_1$ and $\mathbf{p}_1$).

3.2. Solving Local Problem 1

Consider condition (15) as a parametrically defined function describing the motion of the control point P_2 on the drawing plane (depending on the parameters λ_1 and μ_1). Note that for a fixed parameter μ_1, equation (15) describes a parabola along which point P_2 moves. For different values of μ_1, we obtain a family of parabolas. Any point P_2 of any parabola satisfies the smoothness condition (15).

According to the condition of the problem, we must find a point P_2 incident to a predetermined tangent τ_2 (see Fig. 2). The position of the point P_2 running along the parabola (15) is determined (with fixed μ_1 and fixed vectors r1, q1, p1) by the value of the parameter λ_1. Therefore, it is necessary to find a value λ_1 at which the point P_2 falls on the tangent τ_2. In other words, we must find the intersection points of the line τ_2 and the parabola (15).

By decomposing the vector equation (15) along the x, y coordinate axes and assigning an arbitrary value to the parameter μ_1, we obtain the parametric equation of a parabola, where λ_1 plays the role of an independent parameter:

$$\begin{aligned} x_{P2} &= \lambda_1^2(x_1 + x_{Q1} - 2x_{P1}) + 2\lambda_1(x_1 - x_{P1}) + x_1 + 0.5\mu_1(x_1 - x_{P1}) \\ y_{P2} &= \lambda_1^2(y_1 + y_{Q1} - 2y_{P1}) + 2\lambda_1(y_1 - y_{P1}) + y_1 + 0.5\mu_1(y_1 - y_{P1}). \end{aligned} \quad (16)$$

Here (x_1, y_1), (x_{Q1}, y_{Q1}), (x_{P1}, y_{P1}) are the coordinates of the junction point 1 and the control points Q_1, P_1 of the fixed segment 0-1. When changing the parameter μ_1, we obtain a family of parabolas.

The equation of the tangent τ_2 passing through point $2(x_2, y_2)$ and inclined to the x axis at an angle δ_2 has the form:

$$y = tg\delta_2(x - x_2) + y_2 .$$

The control point P_2 of the constructed segment 1-2 must be incident to the case τ_2, therefore, the coordinates x_{P2}, y_{P2} of the point P_2 must satisfy the equation

$$y_{P2} = tg\delta_2(x_{P2} - x_2) + y_2 . \quad (17)$$

Fixing the parameter μ_1 in equation (16) (selecting one parabola from the family of parabolas) and substituting (16) into (17), after algebraic transformations we obtain a quadratic equation with respect to λ_1:

$$\alpha_1\lambda_1^2 + 2\beta_1\lambda_1 + \gamma_1 = 0, \tag{18}$$

where:

$$\begin{aligned} \alpha_1 &= y_1 - 2y_{P1} + y_{Q1} - tg\delta_2(x_1 + x_{Q1} - 2x_{P1}); \\ \beta_1 &= y_1 - y_{P1} - tg\delta_2(x_1 - x_{P1}); \\ \gamma_1 &= y_1 - y_2 - tg\delta_2(x_1 - x_2) + 0.5\beta_1\mu_1. \end{aligned} \tag{19}$$

The value of λ_1 found from (18) ensures that the smoothness condition (15) is fulfilled and that the point P_2 belongs to the previously given tangent τ_2.

In contrast to the generalized smoothness condition (15), in which there is no information about the endpoint 2 of the constructed segment 1-2, nor about the tangent at point 2, all boundary conditions are included in equation (18). Therefore, the value of the parameter λ_1, found from equation (18), allows us to calculate, according to (16), the coordinates of the control point $P_2 \in \tau_2$, which ensures smooth conjugation of segments. The coordinates of the other control point $Q_2 \in \tau_1$ of the segment being constructed are calculated using scalar formulas

$$\begin{aligned} x_{Q2} &= (1+\lambda_1)x_1 - \lambda_1 x_{P1}, \\ y_{Q2} &= (1+\lambda_1)y_1 - \lambda_1 y_{P1}, \end{aligned} \tag{20}$$

equivalent to the vector equation (14).

In equation (18) there are two functionally related variables λ_1 and μ_1. As noted earlier, the parameter μ_1 can be set arbitrarily. Then from (18) we find the value of parameter λ_1 as a function of parameter μ_1:

$$\lambda_1 = \frac{-\beta_1 \pm \sqrt{(D_0 - D_M\mu_1)}}{\alpha_1}, \tag{21}$$

where $D_0 = \beta_1^2 - \alpha_1[y_1 - y_2 - tg\delta_2(x_1 - x_{P1})]$, $D_M = 0.5\beta_1\alpha_1$. The coefficients α_1, β_1 are calculated according to (19).

But it is possible to use a different approach: take the value λ_1 as an independent variable, and assume the parameter μ_1 to be functionally dependent on λ_1. Then from (21) we obtain:

$$\mu_1 = \frac{D_0 - (\alpha_1\lambda_1 + \beta_1)^2}{D_M}. \tag{22}$$

Obviously, equations (21) and (22) are equivalent. Arbitrarily setting the parameter μ_1, we calculate, according to (21), the value of the parameter λ_1; and vice versa, arbitrarily setting the parameter λ_1, we calculate, according to (22), the value of the parameter μ_1. Substituting the pair of μ_1, λ_1 into equations (16) and (20), we calculate the coordinates of the control points μ_1, λ_1 of the constructed segment 1-2. Local problem 1 has been solved.

3.3. Software implementation

An irregular (graphically defined) curve is presented on the computer screen. Points 0, 1, 2, ... are marked on the curve and tangents are fixed at these points τ_0, τ_1, τ_2, ... We must find a composite cubic Bezier curve passing through these points and touching these lines. Following Bezier [2, p. 106], we shall call this an approximating curve, despite the fact that it must pass strictly through the specified points.

Step 1. By moving the control points Q_1, P_1 along the tangents τ_0, τ_1 and drawing the Bezier segment (1) for each combination $\{Q_1, P_1\}$, we achieve a satisfactory coincidence of the Bezier segment with the section 0-1 of the graphically defined curve.

Step 2. We arbitrarily set the parameter λ_1, thereby fixing the control point Q_2 on the case τ_1 (see Fig. 2):

$$\lambda_1 = \frac{w_1^{(2)}}{w_1^{(1)}} = \frac{|1-Q_2|}{|1-P_1|}.$$

Using expressions (20), we calculate the coordinates of the control point Q_2:

$$x_{Q2} = (1+\lambda_1)x_1 - \lambda_1 x_{P1},$$

$$y_{Q2} = (1+\lambda_1)y_1 - \lambda_1 y_{P1}.$$

Note to Step 2. Instead of specifying the numeric value of the parameter λ_1, the constructor can arbitrarily mark the control point Q_2 on the tangent τ_1, and then calculate λ_1.

Step 3. According to (22), we calculate parameter μ_1:

$$\mu_1 = \frac{D_0 - (\alpha_1\lambda_1 + \beta_1)^2}{D_M}.$$

Step 4. We calculate the coordinates of the control point $P_2 \in \tau_2$, the position of which, according to (16), functionally depends on λ_1 and μ_1:

$$x_{P2} = \lambda_1^2(x_1 + x_{Q1} - 2x_{P1}) + 2\lambda_1(x_1 - x_{P1}) + x_1 + 0.5\mu_1(x_1 - x_{P1})$$

$$y_{P2} = \lambda_1^2(y_1 + y_{Q1} - 2y_{P1}) + 2\lambda_1(y_1 - y_{P1}) + y_1 + 0.5\mu_1(y_1 - y_{P1}).$$

Step 5. We calculate a two-dimensional array of points of the Bezier segment 1-2, setting the values of the parameter t in the range $t \in [0, 1]$:

$$x(t) = (1-t)^3 x_1 + 3t(1-t)^2 x_{Q2} + 3t^2(1-t)x_{P2} + t^3 x_2$$

$$y(t) = (1-t)^3 y_1 + 3t(1-t)^2 y_{Q2} + 3t^2(1-t)y_{P2} + t^3 y_2.$$

Result: a Bezier segment passing through points 1, 2 and touching the lines τ_1, τ_2 at these points is obtained. At the junction point 1, the resulting segment has the same radius of curvature as segment 0-1

If satisfactory accuracy of the approximation of section 1-2 is achieved, we proceed to modeling the next section. If the accuracy of the approximation does not satisfy the constructor, we change the position of the point $Q_2 \in \tau_1$ and repeat Steps 2...5. The dialogue continues until the specified accuracy of the approximation is reached in section 1-2.

Note. By specifying different positions of the control point $Q_2 \in \tau_1$, we obtain ∞^1 Bezier segments satisfying the condition of the local problem 1. This is due to the presence of the variable parameter μ_1 in the smoothness condition (15). Neglecting this parameter leads to a loss of variability: for $\mu_1=0$, equation (15) has a unique solution (or has no solution). As a result, the ability to control the shape of the segment being constructed is lost.

Example 1 (Bezier segment shape control). Let the cubic Bezier segment 0-1 be fixed by specifying the control points Q_1, P_1 on the tangents τ_0, τ_1 (Fig. 3). We must find a cubic segment 1-2 that provides smoothness G^2 at the butt point 1 and touching the straight line τ_2 at its endpoint 2.

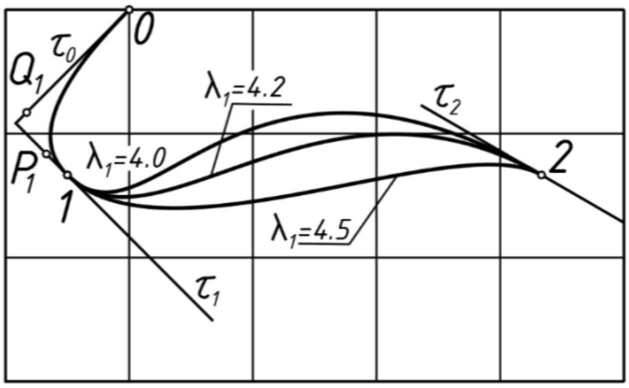

Figure 3. Controlling the shape of a Bezier segment

We assign a value to the parameter λ_1, for example, λ_1=4. Calculate the coordinates (20) of the control point Q_2 of segment 1-2. We calculate the value (22) of the parameter μ_1. Calculate the coordinates (16) of the control point P_2 of segment 1-2. Result: a Bezier segment satisfying the specified boundary conditions is found. At the junction point 1, segments 0-1 and 1-2 have the same radius of curvature.

Assigning different values to the parameter λ_1 (λ_1= 4.0, 4.2, 4.5), we obtain Bezier segments 1-2 of different shapes, but with the same curvature at the junction point 1 (see Fig. 3).

4. Properties of the cubic Bezier curve

Write the equation of the Bezier curve *AB* in projections on the *x, y* axis:

$$
\begin{aligned}
x(t) &= (1-t)^3 x_A + 3t(1-t)^2 x_Q + 3t^2(1-t)x_P + t^3 x_B \\
y(t) &= (1-t)^3 y_A + 3t(1-t)^2 y_Q + 3t^2(1-t)y_P + t^3 y_B,
\end{aligned} \tag{23}
$$

Depending on the position of the point $X = \tau_A \cap \tau_B$ and the control points $Q \in \tau_A$, $P \in \tau_B$, we obtain curves whose shape differs significantly (Fig. 4).

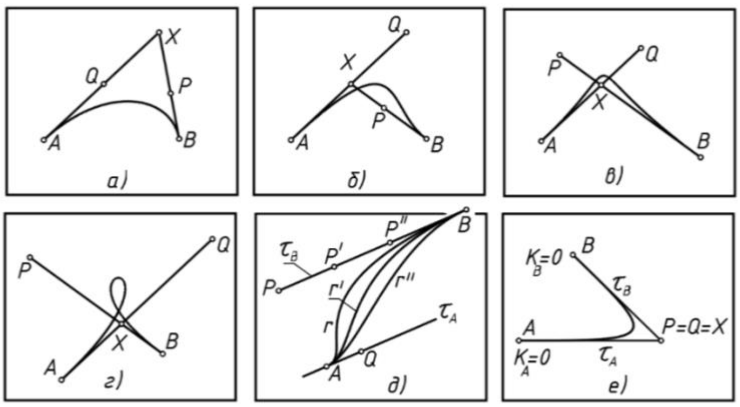

Fig. 4. Types of cubic Bézier curves

Additionally, we consider two special cases: 1) the tangents τ_A, τ_B at the end points of the segment *AB* are parallel to each other; 2) one of the control points (*Q* or *P*) coincides with the point *X* of the intersection of the tangents ($X = \tau_A \cap \tau_B$).

Theorem 1. If the tangents τ_A, τ_B at the end points of the Bezier segment *AB* are parallel to each other, then the curvature of the segment at point *A* is determined only by the position of the control point $Q \in \tau_A$ (it does not depend on the position of the control point $P \in \tau_B$).

Similarly, due to the symmetry of the Bezier segment, its curvature at point B does not depend on the position of the control point $Q \in \tau_A$; it is only determined by the position of point $P \in \tau_B$.

Proof. The curvature of segment (23) at its starting point *A* is calculated by the formula

$$K_A = \frac{\dot{x}_A \ddot{y}_A - \ddot{x}_A \dot{y}_A}{(\dot{x}_A^{\,2} + \dot{y}_A^{\,2})^{3/2}}. \tag{24}$$

We must show that for $\tau_A || \tau_B$, the curvature of K_A does not depend on the position of the point $P(x_P, y_P)$.

Differentiating (23) by the parameter *t* and substituting *t*=0, we obtain:

$$\begin{aligned} \dot{x}_A &= 3(x_Q - x_A);\ \ \ddot{x}_A = 6(x_A - 2x_Q + x_P) \\ \dot{y}_A &= 3(y_Q - y_A);\ \ \ddot{y}_A = 6(y_A - 2y_Q + y_P) \end{aligned} \tag{25}$$

According to (25), the denominator of expression (24) does not depend on the coordinates x_P, y_P of the control point *P*. It is sufficient to show that the numerator of this expression also does not depend on the coordinates of point *P*. Substituting (25) into (24) and performing some algebraic transformations, we obtain an expression for the numerator Ч (without taking into account the constant coefficient):

$$Ч = x_P(y_A - y_Q) - y_P(x_A - x_Q) + x_A y_Q - y_A x_Q \tag{26}$$

The first two terms of this expression, containing the coordinates of the point *P*, are mutually reduced, since, due to the parallelism of the tangents, the equality $y_P / x_P = (y_Q - y_A)/(x_Q - x_A)$ is true.

Thus, neither the denominator nor the numerator of expression (24) depends on the coordinates of point *P*; the theorem is proved. In particular, Bezier segments *r, r', r''* with a common control point *Q* and different control points *P, p', p'',* despite their different shapes, have the same curvature at point A (Fig. 4, e).

Theorem 2. If the vertices *A, Q, P* of the characteristic polyline *AQPB* of the Bezier segment are collinear, then, regardless of the position of point *B*, the curvature of segment *AB* at point *A* is zero (Fig. 5, left). Similarly, if the vertices *B, Q, P* are located collinearly, then, regardless of the position of point *A*, the curvature of the segment *AB* at point *B* is zero (Fig. 5, right).

Proof. Let points *A, Q, P* be collinear (incident to the tangent τ_A). The equation of the tangent τ_A has the form $y = tg\delta_A(x - x_A) + y_A$, therefore:

$$y_Q = tg\delta_A(x_Q - x_A) + y_A, \quad y_P = tg\delta_A(x_P - x_A) + y_A.$$

Substituting y_Q, y_P into (25), we find derivatives $\dot{x}_A, \ddot{x}_A, \dot{y}_A, \ddot{y}_A$. Substituting them into expression (24), we make sure that the numerator of this expression is zero; the second part of the theorem is thus proved.

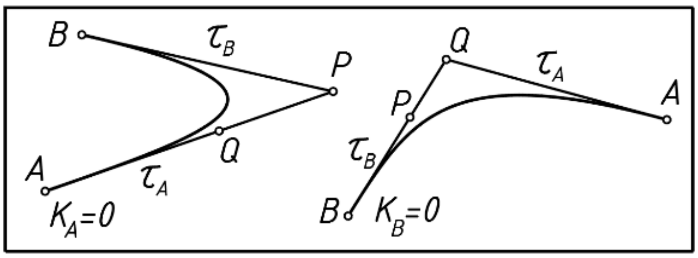

Fig. 5. For Theorem 2

Corollary of Theorem 2. If the control points *Q, P* of characteristic polyline segment *AQPB* Beziers match, the curvature of the segment end points *A, B* is equal to zero (Fig. 4, f).

The corollary of Theorem 2 allows for a G^2-smooth composite curve to be designed with zero curvature in the junction points: it is only necessary to combine the control points of the segments with the points of intersection of tangents at the endpoints of these segments.

Example 2 (smooth conjugation of straight lines with cubic curves). We must draw a G^2-smooth composite curve passing through the points 0, 1, 2, 3, which is touching the given straight lines τ_0, τ_1, τ_2, τ_3 (Fig. 6) at these points. We combine the control points P_1, Q_1 of the first segment 0-1 with the point $X_1 = \tau_0 \cap \tau_1$. Similarly, we combine the control points P_2, Q_2 of the second segment 1-2 with the point $X_2 = \tau_1 \cap \tau_2$, and so on. We obtain a curve consisting of cubic Bezier segments with zero curvature at the junction points, the most tightly of all possible G^2 curves adjacent to its characteristic polyline.

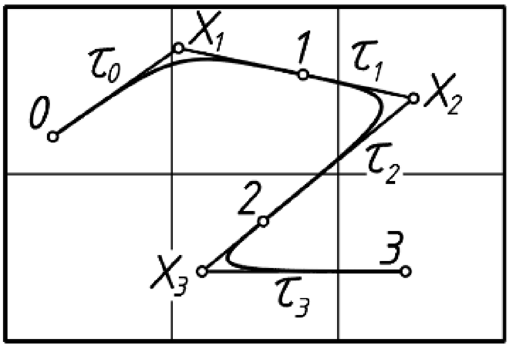

Fig. 6. Composite curve with zero curvature at the butt points

In conclusion of this section we note the "involutional" property of the cubic Bezier curve: when renaming the reference points $A \leftrightarrow B$ and the control points $Q \leftrightarrow P$ simultaneously, the shape of the Bezier segment AB does not change. This follows directly from the consideration of the structure of equation (23), taking into account the fact that for any value of $t \in [0, 1]$, the Bernstein polynomial $(1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3$ is equal to one.

5. Cubic Bezier segment with a given curvature at the end points (solution of Local Problem 2)

Recall the condition of local problem 2 (see paragraph 2): construct a Bezier segment 0-1 by specifying the directions of tangents τ_0, τ_1 and the curvature values K_0, K_1 at the end points 0, 1. We shall look for a solution in the form (1):

$$x(t) = (1-t)^3 x_0 + 3t(1-t)^2 x_Q + 3t^2(1-t) x_P + t^3 x_1$$

$$y(t) = (1-t)^3 y_0 + 3t(1-t)^2 y_Q + 3t^2(1-t) y_P + t^3 y_1.$$

The control points $Q(x_Q, y_Q)$, $P(x_P, y_P)$ must be determined from the incident conditions $Q \in \tau_0$, $P \in \tau_1$ and from the conditions of equality of curvature at the ends of the segment to the values K_0, K_1. The curvature of a plane curve, given explicitly $y = y(x)$, is calculated by the formula

$$K = \frac{y''_x}{(1 + y'^2_x)^{3/2}} \tag{27}$$

The values of $(y'_x)_0 = d_0$ and $(y'_x)_1 = d_1$ are known (equal to the tangents d_0, d_1 of the angles of inclination of the tangents τ_0, τ_1 to the x axis), so the values of the second derivatives $\eta_0 = (y''_x)_0$, $\eta_1 = (y''_x)_1$ at the end points of the segment being constructed can be found from (27).

According to the known rules of differentiation of a complex function, we obtain:

$$d_0 = (y'_x)_0 = \frac{\dot{y}_0}{\dot{x}_0}, \quad d_1 = (y'_x)_1 = \frac{\dot{y}_1}{\dot{x}_1}\ , \tag{28}$$

$$\eta_0 = (y''_x)_0 = \frac{\dot{x}_0\ddot{y}_0 - \ddot{x}_0\dot{y}_0}{(\dot{x}_0)^3}, \quad \eta_1 = (y''_x)_1 = \frac{\dot{x}_1\ddot{y}_1 - \ddot{x}_1\dot{y}_1}{(\dot{x}_1)^3}. \tag{29}$$

Substituting (28) into (29), we obtain:

$$\ddot{y}_0 - d_0\ddot{x}_0 = \eta_0\dot{x}_0^2, \quad \ddot{y}_1 - d_1\ddot{x}_1 = \eta_1\dot{x}_1^2\ . \tag{30}$$

We differentiate (1) by the parameter t:

$$\begin{aligned}
&\dot{x}_0 = 3(x_Q - x_0), \quad \dot{y}_0 = 3(y_Q - y_0),\\
&\dot{x}_1 = 3(x_1 - x_P), \quad \dot{y}_1 = 3(y_1 - y_P),\\
&\ddot{x}_0 = 6(x_0 - 2x_Q + x_P), \quad \ddot{y}_0 = 6(y_0 - 2y_Q + y_P),\\
&\ddot{x}_1 = 6(x_Q - 2x_P + x_1), \quad \ddot{y}_1 = 6(y_Q - 2y_P + y_1).
\end{aligned} \tag{31}$$

Substituting (31) into (30), we obtain:

$$\begin{aligned}
&3\eta_0(x_Q - x_0)^2 = 2(y_0 - 2y_Q + y_P) - 2d_0(x_0 - 2x_Q + x_P),\\
&3\eta_1(x_1 - x_P)^2 = 2(y_Q - 2y_P + y_1) - 2d_1(x_Q - 2x_P + x_1).
\end{aligned} \tag{32}$$

Equations (32) include unknown quantities x_Q, y_Q, x_P, y_P. Given that $Q \in \tau_0$, $P \in \tau_1$, we write:

$$y_Q - y_0 = d_0(x_Q - x_0), \quad y_1 - y_P = d_1(x_1 - x_P)\ . \tag{33}$$

Substituting (33) into (32), we exclude the unknowns y_Q, y_P. After algebraic transformations, we obtain a system of explicit equations with respect to the unknowns x_P, x_Q:

$$\begin{aligned}
&2x_P(d_1 - d_0) = 3\eta_0 x_Q^2 - 6\eta_0 x_0 x_Q + \psi_P,\\
&2x_Q(d_0 - d_1) = 3\eta_1 x_P^2 - 6\eta_1 x_1 x_P + \psi_Q,
\end{aligned} \tag{34}$$

where ψ_P, ψ_Q are constant coefficients:

$$\begin{aligned}
&\psi_P = 3\eta_0 x_0^2 + 2(y_0 - y_1) + 2(d_1x_1 - d_0x_0),\\
&\psi_Q = 3\eta_1 x_1^2 + 2(y_1 - y_0) + 2(d_0x_0 - d_1x_1).
\end{aligned}$$

Solving equations (34) with respect to x_P, x_Q and considering (33), we find the coordinates of the control points $Q(x_Q, y_Q)$, $P(x_P, y_P)$ of the Bezier segment with a given curvature K_0, K_1 at the end points.

The nonlinear system of equations (34) can be solved graphically. Draw parabola $x_P=f(x_Q)$ (the first equation) and parabola $x_Q=g(x_P)$ (the second equation). At the intersection points of the parabolas $f \cap g$, we obtain the values of the unknown x_Q, x_P. According to (33), we calculate the values of y_Q, y_P. The control points are determined.

Note. Intersecting at four points, the parabolas give four solutions to the system of equations (34) (four variants of the characteristic polyline). All four variants induce Bezier segments with the same curvature modulus $|K_0|$ at the starting point of the segment, and the same curvature modulus $|K_1|$ at the end point. At the same time, only one option corresponds to the predetermined curvature sign at the ends of the segment being constructed.

Example 3 (conjugation of two circles). We must construct a cubic Bezier curve smoothly connecting two given circles with radii R_A, R_B. The conjugation points A, B are indicated on the circles (Fig. 7). Having drawn, according to (34), the parabolas $x_P=f(x_Q)$ and $x_Q=g(x_P)$, we mark their intersection points 1, 2, 3, 4. We obtain four solutions of the system of equations (34). Each solution corresponds to a cubic Bezier segment with specified radii of curvature R_A, R_B at the boundary points A, B. Figure 7 shows the construction of control points Q_1, P_1 of the cubic segment No. 1 corresponding to point 1 of the intersection of parabolas.

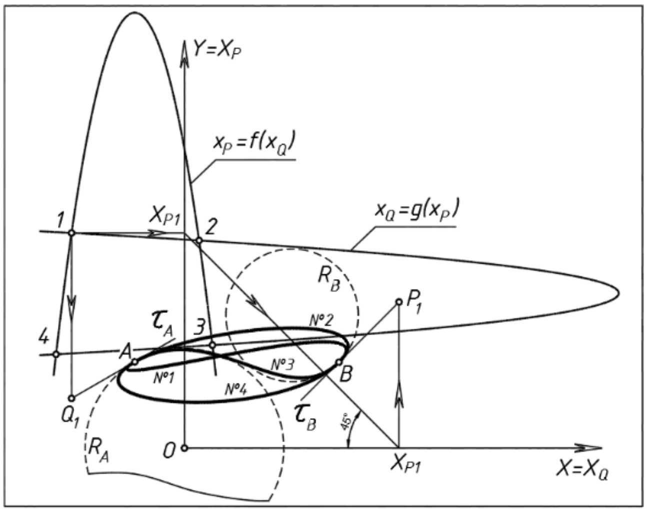

Fig. 7. Conjugation of two circles

Example 4 (conjugation of a straight line and a circle). The direction $\mathbf{T}_A$ of a straight line passing through point A is indicated. The point B is indicated on the circle R_B (Fig. 8). It is required to construct a transition curve AB, smoothly (without curvature jumps) connecting a straight line and a circle.

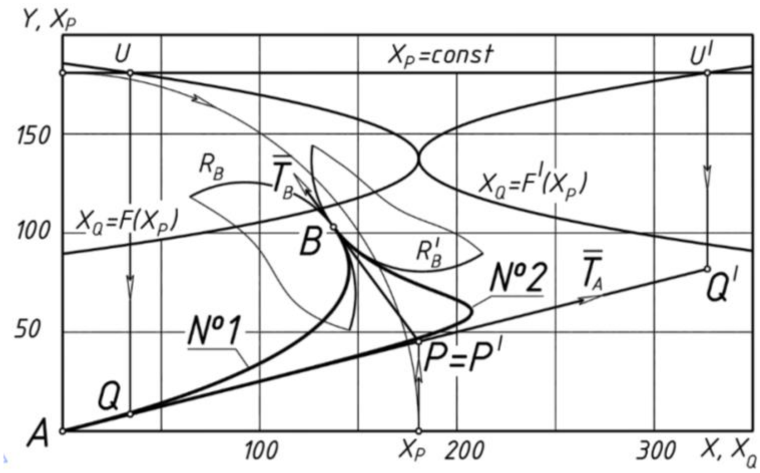

Figure 8. Conjugate a circle and a straight line

The curvature of the transition curve at point A must be zero. Therefore, according to Theorem 2, the control point P of the desired curve must coincide with the intersection point of the directions $\mathbf{T}_A$, $\mathbf{T}_B$. Here $\mathbf{T}_B$ is a vector touching the circle R_B at point B.

Substituting $\eta_0=\eta_A=0$ into the first equation (34), we obtain a degenerate parabola x_P=const. Substituting the calculated value of $\eta_1=|\eta_B|$ into the second equation (34), we obtain two parabolas $x_Q=F(x_P)$ and $x_Q=F'(x_P)$. We mark the points U, U' of the intersection of the parabolas with a straight line x_P=const. We obtain characteristic polylines *(A-Q-P-B)* and *(A-Q'-P'-B)*, which correspond to Bezier segments No. 1 and No. 2. The task condition is satisfied by segment #1.

Example 5 (closed G^2 is a smooth contour). We must form a smooth, closed contour touching the sides of the square at points 0, 1, 2, 3. The radius of curvature is set at point 0.

The problem has many solutions. Using a composite cubic Bezier curve, we can obtain both symmetric (Fig. 9, a, b) and asymmetric (Fig. 9, c, d) closed G^2-smooth contours satisfying the conditions of the problem. When constructing the contours, algorithms for solving local problems 1 and 2 were used.

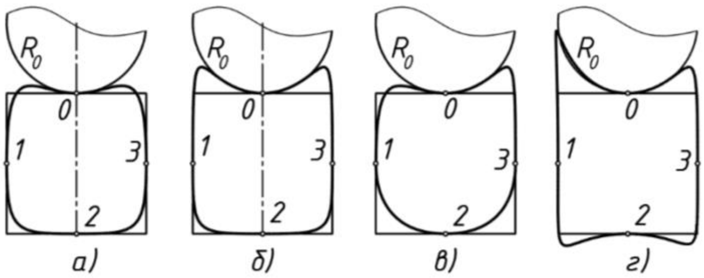

Figure 9. Closed G2-smooth contour (options)

6. Simulation of a physical spline (experiment)

A physical spline is a line formed by the axis of an elastic rod passing through predetermined points. It is assumed that the dimensions of the cross-section of the rod are very small compared to the length and radius of curvature of its axis. An example of such a spline is an elastic metal ruler. Passing through points set on the plane, the ruler naturally acquires a shape with minimum energy of internal stresses and minimum average curvature. The theoretical equation of a physical spline can be found only under the condition of small deflections (small deviations from a straight line). In this case, the physical spline is satisfactorily described by a composite piecewise cubic polynomial curve of the second degree of smoothness [10]. For large deflections, the solution becomes fundamentally more complicated. According to [6], it reduces to a variational problem that has no elementary solution. Therefore, it is advisable to model a physical spline with large deflections experimentally, followed by approximation of the resulting curve.

The simplest physical spline. A physical spline passes through points *A, B, C* (Fig. 10, left). A "three-point" spline with free ends can be called the simplest physical spline. We must find an analytical function that gives a satisfactory approximation to the elastic line of the simplest spline.

The desired function must satisfy three groups of local conditions: incidence to the reference points *A, B, C*; tangency of the lines τ_A, τ_B, τ_C; and zero curvature at points *A, C*. The problem cannot be solved using a standard NURBS curve that does not account for the pre-defined local geometric characteristics of the simulated line. Euler elastics are also inapplicable, since there is no axial force acting on the elastic element [5].

We shall look for a solution in the form of a composite G^2-smooth cubic Bezier curve. We divide the elastic line into sections *AB* and *BC*, each of which is replaced by a cubic Bezier segment. To ensure zero curvature at point *A*, we combine the control point P_1 of segment *AB* with the intersection point of tangents τ_A, τ_B (see Theorem 2). By moving the control point Q_1 along the tangent τ_A, we achieve the required accuracy of the approximation of the section *AB*. To ensure zero curvature at point *C*, we combine the control point Q_2 with the intersection point of the tangents τ_B, τ_C (see Theorem 2). Calculate the value of the parameter λ_B (see paragraph 3.3):

$$\lambda_B = \frac{w_B^{(2)}}{w_B^{(1)}} = \frac{|B - Q_2|}{|B - P_1|} .$$

Through formula (22) we find the value of the parameter μ, at which the condition (18) of a smooth connection of the segments *AB* and *DC* is fulfilled. Substituting the found values λ_B and μ in (16), we obtain the coordinates of the control point P_2. The control points of the *BC* segment are fully defined. The composite cubic curve *AB*+*BC* satisfies all boundary conditions. The approximation error does not exceed 1.5% (Fig. 10, right).

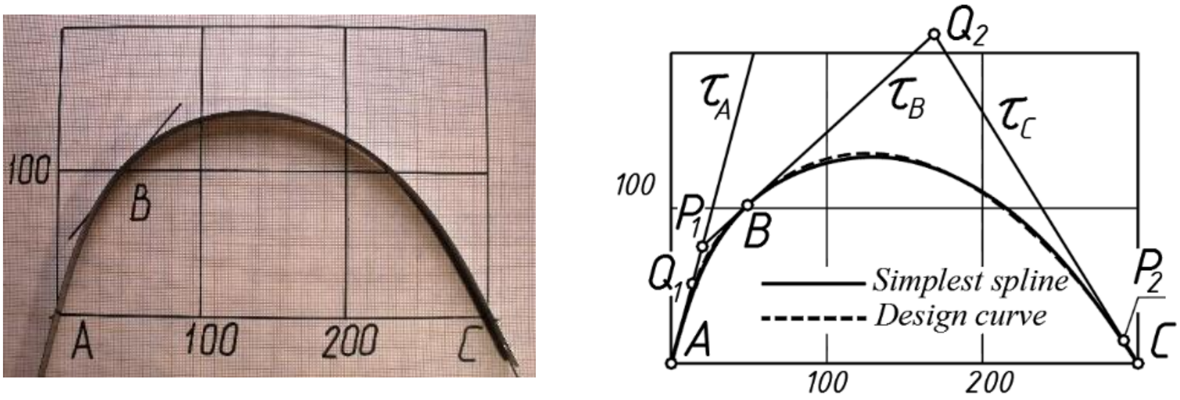

Figure 10. The simplest physical spline: photo (left) and approximation (right)

General physical spline. An elastic element with free ends passes through the reference points 0, 1, ..., 4. At the reference points we mark the tangents τ_0, ..., τ_4 (Fig. 11, left). We must find a G^2-smooth approximating function passing through points 0, 1, ..., 4 and touching the lines τ_0, ..., τ_4. The curvature of the approximating function at the end points 0 and 4 should be zero. Let us look for a solution in the form of a curve composed of four cubic Bezier segments.

The first segment. The control point P_1 of the first segment 0-1 is combined with the intersection point of the tangents τ_0, τ_1. Result: the curvature of segment 0-1 at the starting point is zero (see Theorem 2). By moving the control point Q_1 along the tangent τ_0, we achieve a satisfactory approximation of the first section of the physical spline.

The second segment. Specifying the control point $Q_2 \in \tau_1$, we find the control point $P_2 \in \tau_2$. The position of point P_2 functionally depends on the position of point Q_2 (see paragraph 3.3). By moving the point Q_2 along the tangent τ_1, we achieve a satisfactory approximation of the second section of the physical spline.

The third segment. Specifying the control point $Q_3 \in \tau_2$, we find the control point $P_3 \in \tau_3$. The position of point P_3 functionally depends on the position of point Q_3. By moving the point Q_3 along the tangent τ_2, we achieve a satisfactory approximation of the third section of the physical spline.

The fourth segment. We combine the control point Q_4 with the intersection point of the tangents τ_3, τ_4. Result: the curvature of the Bezier segment 3-4 at the endpoint 4 is zero (see Theorem 2). The curvature at the ends of the fourth segment is fixed, so its shape cannot be controlled (see paragraph 5). Nevertheless, the Bezier segment 3-4 satisfactorily approximates the fourth section of the physical spline. The approximation error is less than 2% (Fig. 11, right).

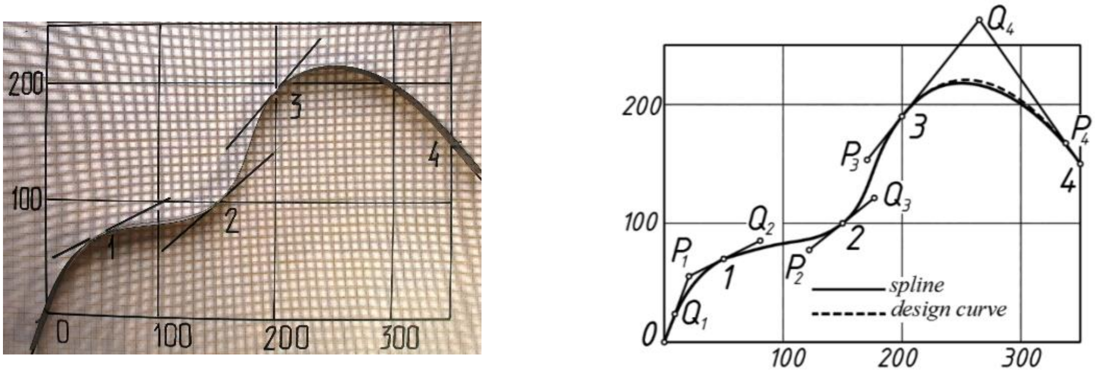

Figure 11. General view physical spline approximation (photo and drawing)

7. Conclusion

Flat, graphically defined irregular curves are found in various engineering problems. To use such a curve in the design process, it must be approximated with the necessary accuracy through a relatively simple analytical function (or a set of such functions interconnected with a certain degree of smoothness). A compromise between accuracy and simplicity of mathematical description can be achieved through the use of composite cubic Bezier curves. The practical application of such curves is complicated by the absence in the technical literature of algorithms for calculating the coordinates of the control points of Bezier segments which account for the pre-set local characteristics of the curve being constructed (such as tangents and curvature at the nodal and end points).

The article proposes an algebraic algorithm (paragraph 3.2, *Local problem* 1) and a program module (paragraph 3.3, software implementation) that can be used to determine the coordinates of the control points of the connected Bezier segments and control the shape of the segments without disturbing the order of smoothness G^2 at the junction points. It is shown that the solution of the smooth docking problem reduces to the solution of the quadratic equation (18).

A graphoanalytic algorithm for constructing a planar cubic Bezier segment given by the values of the first and second derivatives at the ends of the segment (paragraph 5, *Local problem* 2) is compiled. The search for control points of such a segment is reduced to solving a system of two quadratic equations (34) or to determining the coordinates of the intersection points of two drawn parabolas. The developed algorithms are used to approximate the experimentally obtained physical spline. The approximation error was less than 2%

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