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A Practical System for Defining Bézier Surfaces

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Abstract

This paper considers various options for modeling complex-shaped surfaces using cubic Bézier curves as generatrices. The generatrix grid algorithm known from P. Bézier’s works allows defining the patch passing through a pre-defined spatial closed contour. In the general case, the generatrix grid of a Bézier patch is a set of non-planar spatial lines. However, such a solution is not technologically feasible. The paper demonstrates the ability to generate a quadrangular patch of a Bézier surface containing two families of plane generatrices. Bézier surfaces resting on a plane frame formed by the boundary curves of arbitrary shapes can be also formed.

The smoothness conditions proposed by P. Bézier in 1972 are used to obtain a G1-smooth composite Bézier surface. These conditions ensure that there is a break in the generatrices on the line connecting the patches, while the surface itself remains G1-smooth. Certain properties of the Bézier curve allow forming a G2-smooth conjugation between two planes.

The paper discusses the case where Coons and Bézier patches (surface portions) coincide and pass through a fixed quadrilateral contour and provides examples of how to visualize Bézier surfaces under different boundary conditions using computer graphics.

Keywords: cubic Bézier curve, plane generatrices, affine correspondence, composite surface, smoothness, conjugation of planes.

1. Introduction

This paper examines the practical aspects of forming a Bézier surface on an arbitrary spatial triangular or quadrilateral contour. P. Bézier’s seminal work [1] introduced the concept of geometric modeling of composite curves and surfaces, as well as the conditions for the smooth conjugation of curve segments and patches. In a book by French authors, Chapter 4 discusses P. Bézier’s geometric method. A translation of this book can be found in [2]. Monographs and textbooks [3…11] explore various mathematical and geometric aspects of CAGD.

Historically, the Bézier method was preceded by the Coons method [12], which implies that a patch is formed as a superposition of three ruled surfaces. This ensures the computational simplicity of the method. In some cases, Bézier and Coons patches are identical (see Section 8 hereof).

The monograph [13] by English authors dealing with the practical aspects of constructing lines and surfaces is best suited for a prior acquaintance with computational geometry methods. The authors of [14] consider the geometric modeling of three-dimensional objects with a focus on solid modeling. A brief overview of methods for creating composite curves and surfaces can be found in the reference manual [15]. Constructive algorithms for generating composite G2-smooth cubic Bézier curves in space and on the plane are presented in the articles [16…18].

A quadrangular patch of a cubic Bézier surface is defined by specifying sixteen vertices 00, 01, ..., 32, 33 of the characteristic polyhedron [1, 2, 3]. Each point of the surface is associated with a radius vector $\mathbf{r}(u,v)$. For example, corner point 03 corresponds to the radius vector $\mathbf{r}_{03}$ (Fig. 1).

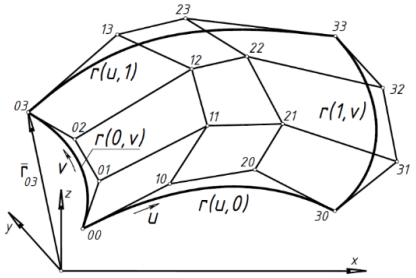


Fig. 1. Characteristic polyhedron and boundary curves of a Bézier patch

The patch equation has the form

$$\mathbf{r}(u,v)=\sum_{i=0}^{3}[\sum_{j=0}^{3}\mathbf{r}_{i,j}g_j(v)]g_i(u),\quad u,v\in[0,1], \tag{1}$$

where $g_k(t)=\dfrac{3!}{k!(3-k)!}t^k(1-t)^{3-k},\quad k=0,1,2,3.$

If we expand (1) and group the terms containing the v parameter, we obtain:

$$\mathbf{r}(u,v)=(1-u)^3\mathbf{r}(0,v)+3u(1-u)^2\mathbf{q}(v)+3u^2(1-u)\mathbf{p}(v)+u^3\mathbf{r}(1,v)\,, \tag{2}$$

where

$$\begin{aligned}
\mathbf{r}(0,v)&=(1-v)^3\mathbf{r}_{00}+3v(1-v)^2\mathbf{r}_{01}+3v^2(1-v)\mathbf{r}_{02}+v^3\mathbf{r}_{03},\\
\mathbf{q}(v)&=(1-v)^3\mathbf{r}_{10}+3v(1-v)^2\mathbf{r}_{11}+3v^2(1-v)\mathbf{r}_{12}+v^3\mathbf{r}_{13},\\
\mathbf{p}(v)&=(1-v)^3\mathbf{r}_{20}+3v(1-v)^2\mathbf{r}_{21}+3v^2(1-v)\mathbf{r}_{22}+v^3\mathbf{r}_{23},\\
\mathbf{r}(1,v)&=(1-v)^3\mathbf{r}_{30}+3v(1-v)^2\mathbf{r}_{31}+3v^2(1-v)\mathbf{r}_{32}+v^3r_{33}.
\end{aligned} \tag{3}$$

Here $\mathbf{r}(0,v)$ and $\mathbf{r}(1,v)$ are the left and right boundaries of the patch (see Fig. 1). The curves $\mathbf{q}(v)$ and $\mathbf{p}(v)$ that do not lie on the constructed surface are called control curves, since they carry rows of points controlling the shape of longitudinal generatrices (Fig. 2). Equation (2) describes the family of longitudinal generatrices of the surface (1) for different values of the v parameter.

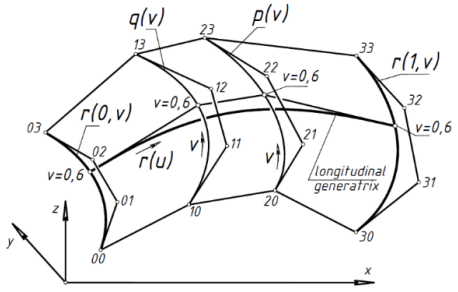


Fig. 2. Construction of a longitudinal generatrix

The terms containing the u parameter should be grouped to construct a family of transverse generatrices in equation (1). We obtain:

$$\mathbf{r}(u,v)=(1-v)^3\mathbf{r}(u,0)+3v(1-v)^2\mathbf{q}(u)+3v^2(1-v)\mathbf{p}(u)+v^3\mathbf{r}(u,1), \tag{4}$$

where

$$
\begin{aligned}
\mathbf{r}(u,0) &= (1-u)^3\mathbf{r}_{00} + 3u(1-u)^2\mathbf{r}_{10} + 3u^2(1-u)\mathbf{r}_{20} + u^3\mathbf{r}_{30},\\
\mathbf{q}(u) &= (1-u)^3\mathbf{r}_{01} + 3u(1-u)^2\mathbf{r}_{11} + 3u^2(1-u)\mathbf{r}_{21} + u^3\mathbf{r}_{31},\\
\mathbf{p}(u) &= (1-u)^3\mathbf{r}_{02} + 3u(1-v)^2\mathbf{r}_{12} + 3u^2(1-u)\mathbf{r}_{22} + u^3\mathbf{r}_{32},\\
\mathbf{r}(u,1) &= (1-u)^3\mathbf{r}_{03} + 3u(1-u)^2\mathbf{r}_{13} + 3u^2(1-u)\mathbf{r}_{23} + u^3r_{33}.
\end{aligned}
\tag{5}
$$

Here, $\mathbf{r}(u,0)$ and $\mathbf{r}(u,1)$ are the lower and upper boundaries of the patch. The curves $\mathbf{q}(u)$ and $\mathbf{p}(u)$, which do not lie on the surface, carry rows of points controlling the shape of the transverse generatrices $\mathbf{r}(v)$ (Fig. 3). Equation (4) describes the family of transverse generatrices of the surface (1) for different values of the u parameter.

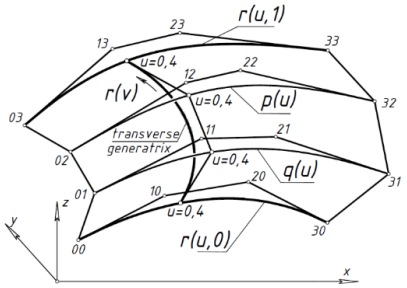


Fig. 3. Construction of a transverse generatrix

2. Problem statement and solution algorithm

In three-dimensional space, we define Bézier curves $\mathbf{r}(0,v)$, $\mathbf{r}(1,v)$, $\mathbf{r}(u,0)$, $\mathbf{r}(u,1)$ forming a closed quadrangular contour (see Fig. 1). Our goal is to determine the Bézier patch 'spanned' by the given contour. The surface is defined if there is an algorithm for constructing a set of its longitudinal and transverse generatrices.

Generatrix construction algorithm. We will assume that the boundary curves of the Bézier patch and vertices 11, 12, 21, 22 of the characteristic polyhedron are fixed (see Fig. 1).

Step 1 (construction of longitudinal generatrices). According to (3), we construct the control curves $\mathbf{q}(v)$ and $\mathbf{p}(v)$. Then we fix a value $v=v_0$, for example, $v_0=0.6$. We mark the points $\mathbf{r}(0,v_0)$ and $\mathbf{r}(1,v_0)$ on the boundary curves $\mathbf{r}(0,v)$, $\mathbf{r}(1,v)$. We mark the points $\mathbf{q}(v_0)$, $\mathbf{p}(v_0)$ on the control curves $\mathbf{q}(v)$, $\mathbf{p}(v)$. We obtain four points that define the characteristic broken line of the Bézier curve $\mathbf{r}(u)$. This curve is the longitudinal generatrix of the Bézier patch (see Fig. 2). We assign different values to the v parameter to generate a family of longitudinal generatrices of the desired patch.

Step 2 (construction of transverse generatrices). According to (5), we construct the control curves $\mathbf{q}(u)$, $\mathbf{p}(u)$. Then we fix a value $u=u_0$, for example, $u_0=0.4$. We mark the points $\mathbf{r}(u_0,0)$ and $\mathbf{r}(u_0,1)$ on the boundary curves $\mathbf{r}(u,0)$, $\mathbf{r}(u,1)$. We mark the points $\mathbf{q}(u_0)$, $\mathbf{p}(u_0)$ on the control curves $\mathbf{q}(u)$, $\mathbf{p}(u)$. We obtain four points defining the characteristic broken line of the Bézier curve $\mathbf{r}(v)$. This curve is the transverse generatrix of the Bézier patch (see Fig. 3). We assign different values to the u parameter to generate a family of transverse generatrices of the desired patch.

This algorithm works for any boundary Bézier curves. These curves can be plane or spatial; they can degenerate to a straight line or a point. For example, Fig. 4 shows a Bézier surface spanned by an arbitrarily given curvilinear spatial contour.

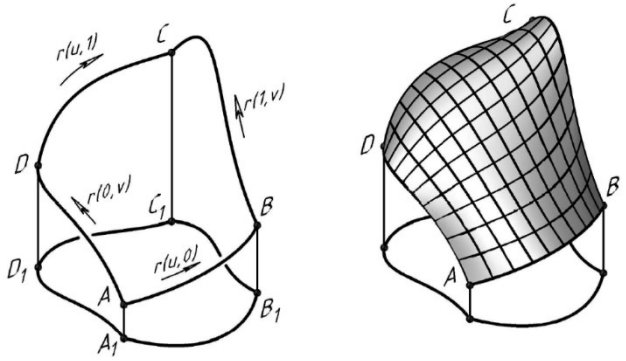
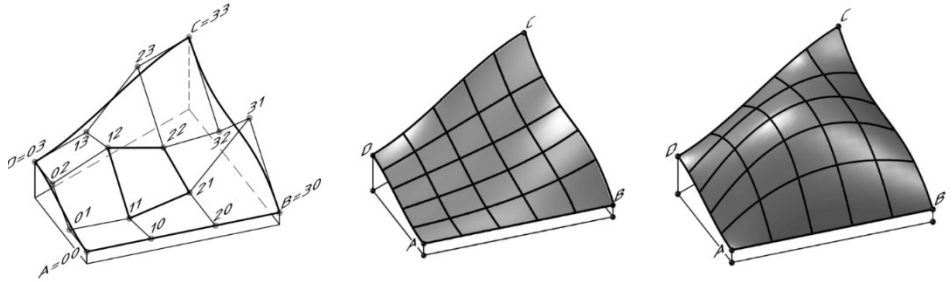


Fig. 4. Bézier surface on a spatial contour

Figure 5, a shows a characteristic polyhedron with a rectilinear boundary *AB*. We change the position of control points 11, 12, 21, 22 to obtain a concave (Fig. 5, b) or convex (Fig. 5, c) patch of the Bézier surface.

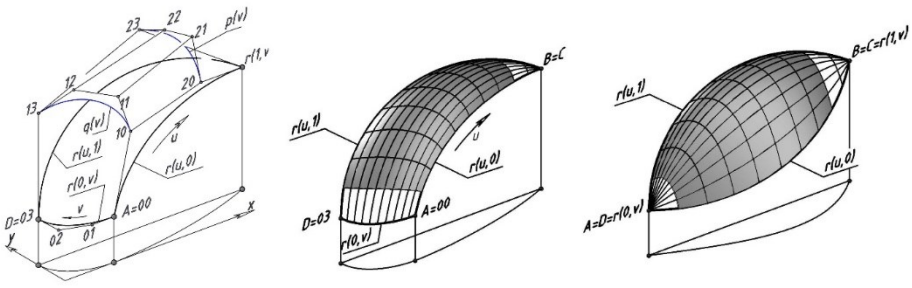


a) b) c)

Fig. 5. Bézier patch: a – characteristic polyhedron; b – concave surface; c – convex surface

3. Degenerate Bézier patches

If one of the boundaries of a quadrangular Bézier patch degenerates into a point, we obtain a triangular patch. Let us assume for the purposes of definiteness that the right boundary Bézier curve $\mathbf{r}(1,v)$ defined by characteristic broken line 30, 31, 32, 33 has degenerated into point $B=C$ (Fig. 6, a). Characteristic broken lines 10-11-12-13 and 20-21-22-23 allow drawing the Bézier control curves $\mathbf{q}(v)$ and $\mathbf{p}(v)$ (marked in blue). We fix an arbitrary value of the $v=v_0$ parameter. We mark the point $\mathbf{r}(0,v_0)$ on the left boundary curve *AD*. The right boundary curve has degenerated into point $B=C$; therefore, $\mathbf{r}(1,v_0)=B=C$. We mark points $\mathbf{q}(v_0)$, $\mathbf{p}(v_0)$ on the control curves. We obtain four points $\mathbf{r}(0,v_0)$, $\mathbf{q}(v_0)$, $\mathbf{p}(v_0)$, $\mathbf{r}(1,v_0)=B=C$ defining the characteristic broken line of the Bézier curve $\mathbf{r}(u)$. This curve is the longitudinal generatrix of the Bézier patch at $v=v_0$. We assign different values of the v parameter to find a family of longitudinal generatrices of the desired patch. Similarly, we fix different values of the u parameter to find a family of transverse generatrices.



a) b) c)

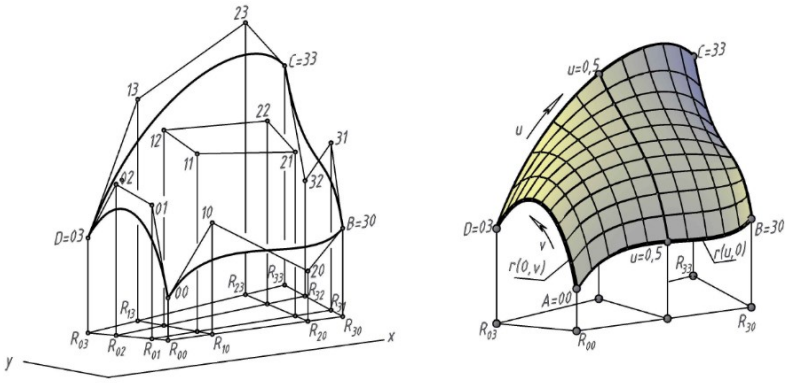
Fig. 6. Triangular Bézier patches: a – characteristic polyhedron; b – triangular patch; c – patch with two degenerate boundary curves

The generatrix construction algorithm described in Section 2 is completely preserved in the case where two opposite boundaries of a quadrilateral contour have degenerated into points. Figure 6, c shows a Bézier patch with degenerate boundaries $\mathbf{r}(0,v)=A=D$ and $\mathrm{r}(1,v)=B=C$.

4. Quadrangular Bézier patch with plane generatrices

If the vertices of the characteristic polyhedron are assigned arbitrarily, the generatrix grid of the Bézier patch is a set of spatial curves with nonzero torsion. This solution is not technologically feasible. Let us demonstrate the ability to generate a quadrangular patch of the Bézier surface containing two families of plane generatrices. To this end, we impose certain conditions on the selection of the vertices of the characteristic polyhedron.

Consider the corner points $A=00$, $B=30$, $C=33$, $D=03$ for a quadrangular Bézier patch. The orthogonal projection of the patch onto the xy plane is an irregular quadrilateral $R_{00}R_{30}R_{33}R_{03}$ (Fig. 7, a).



a) b)

Fig. 7. Bézier patch with plane generatrices: a – characteristic polyhedron with affinely corresponding opposite sides of the plan; b – visualization

Based on design considerations, we define two adjacent boundaries of the patch, for example, $\mathbf{r}(0,v)$ and $\mathbf{r}(u,0)$. Thus, we fix eight vertices of the characteristic polyhedron of the Bézier patch (four corner points and two pairs of control points of the adjacent boundaries of the patch). To define the surface, we need to define eight more vertices. If we set these vertices randomly, we will obtain a grid of spatial generatrices with nonzero torsion.

To obtain the plane generatrices of the Bézier patch, we need to ensure an affine correspondence between the opposite sides of the plan. An affine correspondence between the left and right sides is achieved when the following ratios are equal

$$\frac{|R_{30}-R_{31}|}{|R_{30}-R_{33}|}=\frac{|R_{00}-R_{01}|}{|R_{00}-R_{03}|},\quad \frac{|R_{30}-R_{32}|}{|R_{30}-R_{33}|}=\frac{|R_{00}-R_{02}|}{|R_{00}-R_{03}|}. \tag{6}$$

An affine correspondence between the front and back sides of the plan is achieved when the following ratios are equal

$$\frac{|R_{03}-R_{13}|}{|R_{03}-R_{33}|}=\frac{|R_{00}-R_{10}|}{|R_{00}-R_{30}|},\quad \frac{|R_{03}-R_{23}|}{|R_{03}-R_{33}|}=\frac{|R_{00}-R_{20}|}{|R_{00}-R_{03}|}. \tag{7}$$

Conditions (6), (7) are satisfied with the parallel projection of one side of the plan onto its opposite side. These conditions determine the positions of the horizontal projections of vertices 13, 23, 31, 32 of the characteristic polyhedron. There are no restrictions on the altitude position of these vertices, which allows varying the shapes of the boundary curves $\mathbf{r}(1,v)$ and $\mathbf{r}(u,1)$.

The horizontal projections of internal control vertices 11, 12, 21, 22 are determined at the intersection of the segments connecting the affine-corresponding points of the opposite sides of the plan. The heights of these vertices can be set arbitrarily.

Figure. 7, b shows a Bézier patch generated using the specified conditions. The plane generatrix corresponding to the value u=0.5 is highlighted in the figure. The resulting convex-concave surface carries two families of plane generatrices lying in vertical planes.

5. Bézier patch on a plane frame

We need to span a Bézier surface by a plane quadrangular frame. The boundary Bézier curves degenerate into four segments (the sides of the frame). Control points (01, 02), (10, 20), (31, 32), (13, 23) on the sides of the frame can be assigned arbitrarily (Fig. 8, a). The longitudinal and transverse generatrices of the surface are determined according to the algorithm described in Section 2, resulting in a surface with non-plane generatrices (Fig. 8, b, c).

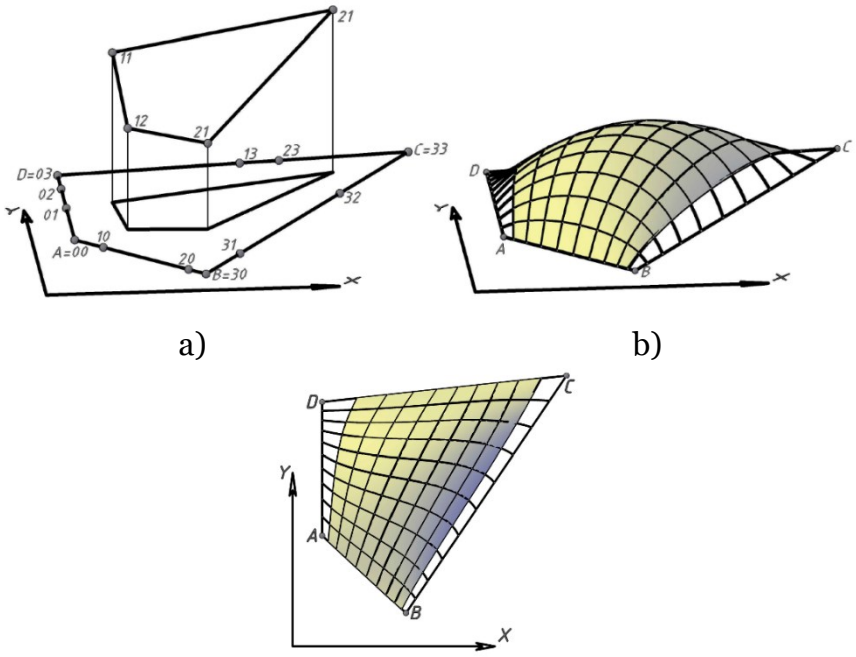
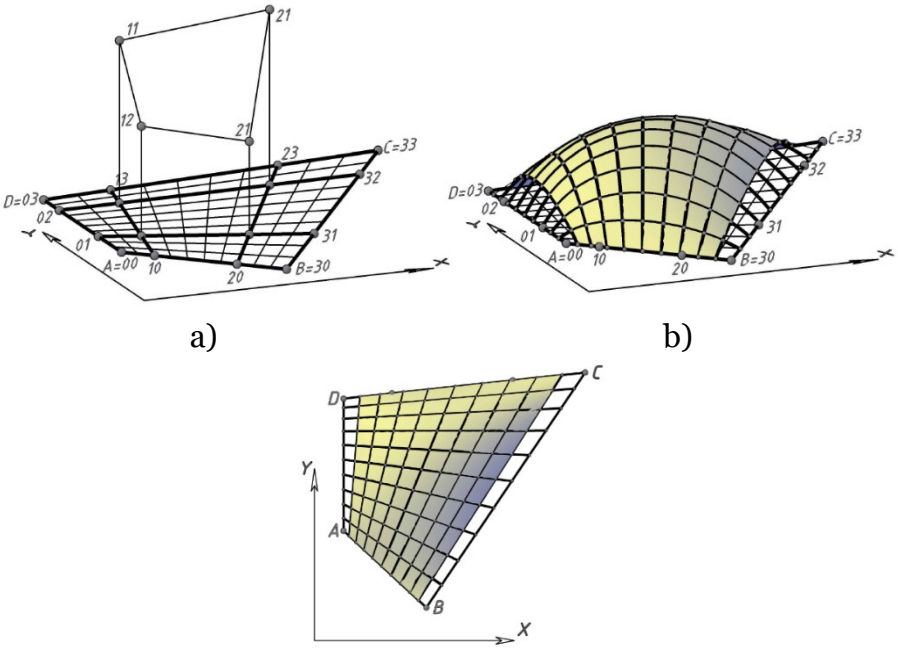


Fig. 8. Bézier patch on a plane frame: a) – arbitrarily specified control points; b – axonometry; c – plan

To obtain the plane generatrices of a Bézier patch, we should ensure the affine correspondence between the segments on the opposite sides of the frame. The control points of adjacent sides, for example, points (01, 02) and (10, 20), can be specified arbitrarily. Control points (31, 32) and (13, 23) are determined from the affine correspondence between the opposite sides (Fig. 9, a). The horizontal projections of vertices 11, 12, 21, 22 of the characteristic polyhedron are defined at the intersection of segments connecting the affine-corresponding points of the opposite sides of the frame. The height of these vertices can be specified arbitrarily. We obtain a Bézier patch on a plane quadrangular frame. This surface carries two families of plane generatrices lying in vertical planes (Fig. 9, b, c).

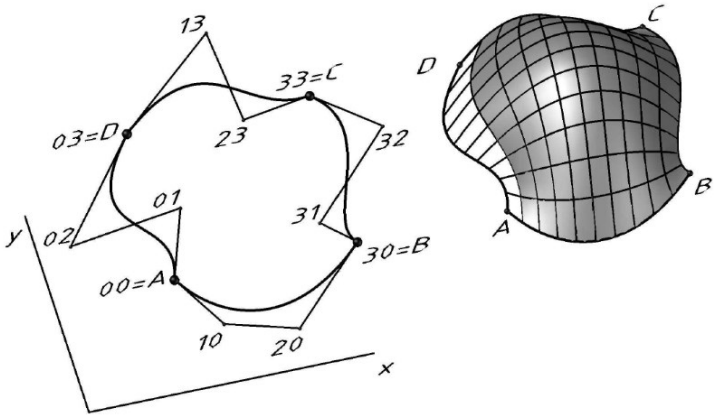


a) b)

c)

Fig. 9. Bézier patch with plane generatrices: a) – control points; b – axonometry; c – plan

Let the frame be formed by plane Bézier curves, rather than rectilinear segments. The twelve vertices of the characteristic polyhedron are fixed (Fig. 10, a). Internal control vertices 13, 23, 31, 32 can be specified arbitrarily. Any change in the position of these vertices changes the shape of the Bézier patch. The algorithm described in Section 2 allows constructing an arbitrarily dense grid of longitudinal and transverse generatrices (Fig. 10, b). Plane generatrices cannot be obtained.



a) b)

Fig. 10. Bézier patch on the plane frame of arbitrary shape: a) – the frame is formed by Bézier curves; b – visualization

6. Smooth connection of Bézier patches

Let us consider the conditions for a G1-smooth connection of two Bézier patches (Fig. 11). According to [13, p. 195], the normal vectors to patches $ABCD=\mathbf{r}^{(1)}(u,v)$ и $BMNC=\mathbf{r}^{(2)}(u,v)$ should coincide at any point on the line of the connecting seam BC:

$$\mathbf{r}_u^{(2)}(0,v)\times\mathbf{r}_v^{(2)}(0,v)=\lambda\mathbf{r}_u^{(1)}(1,v)\times\mathbf{r}_v^{(1)}(1,v), \quad (8)$$

where λ is an arbitrary number. Here the superscript indicates the patch number and the subscript indicates the variable used for differentiation. The constant factor λ means that the

modulus of the normal vector to patches 1 and 2 may be discontinuous when passing through the seam line.

Condition (8) is satisfied if [13, p. 197]:

$$\mathbf{r}_u^{(2)}(0,v) = \lambda \mathbf{r}_u^{(1)}(1,v) + (\mu_0 + \mu_1 v)\mathbf{r}_v^{(1)}(1,v), \tag{9}$$

where μ_0 and μ_1 are arbitrary numbers.

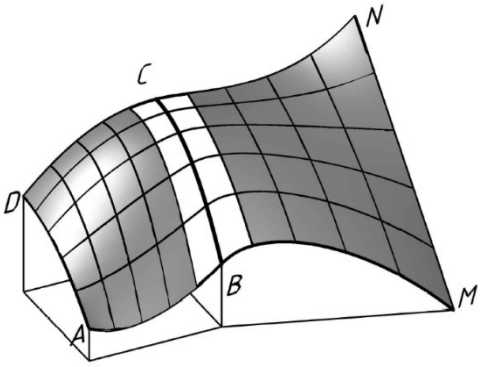
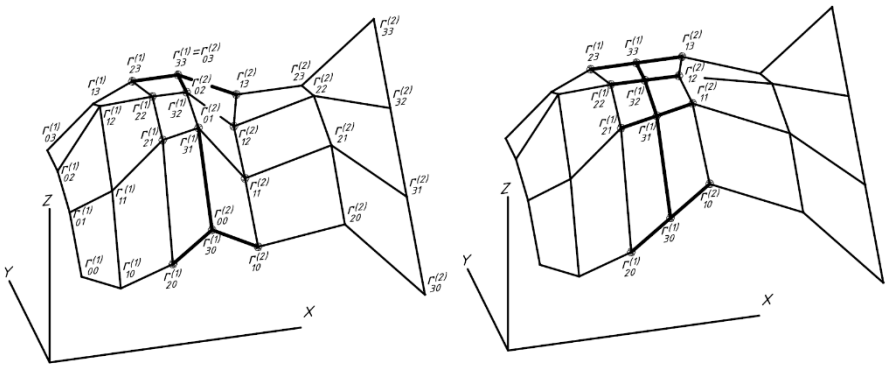


Fig. 11. Smooth connection of Bézier portions

If we differentiate the equations of the patches $\mathbf{r}^{(1)}(u,v)$, $\mathbf{r}^{(2)}(u,v)$ with respect to the u and v parameters and equate the coefficients at the same powers of v, we obtain the well-known [13, p. 198] smoothness conditions:

$$\begin{aligned}
\mathbf{r}_{10}^{(2)} &= \mathbf{r}_{30}^{(1)} + \lambda(\mathbf{r}_{30}^{(1)} - \mathbf{r}_{20}^{(1)}) + \mu_0(\mathbf{r}_{31}^{(1)} - \mathbf{r}_{30}^{(1)}),\\
\mathbf{r}_{11}^{(2)} &= \mathbf{r}_{31}^{(1)} + \lambda(\mathbf{r}_{31}^{(1)} - \mathbf{r}_{21}^{(1)}) + \frac{1}{3}\mu_0(2\mathbf{r}_{32}^{(1)} - \mathbf{r}_{31}^{(1)} - \mathbf{r}_{30}^{(1)}) + \frac{1}{3}\mu_1(\mathbf{r}_{31}^{(1)} - \mathbf{r}_{30}^{(1)}),\\
\mathbf{r}_{12}^{(2)} &= \mathbf{r}_{32}^{(1)} + \lambda(\mathbf{r}_{32}^{(1)} - \mathbf{r}_{22}^{(1)}) + \frac{1}{3}\mu_0(\mathbf{r}_{33}^{(1)} + \mathbf{r}_{32}^{(1)} - 2\mathbf{r}_{31}^{(1)}) + \frac{2}{3}\mu_1(\mathbf{r}_{32}^{(1)} - \mathbf{r}_{31}^{(1)}),\\
\mathbf{r}_{13}^{(2)} &= \mathbf{r}_{33}^{(1)} + \lambda(\mathbf{r}_{33}^{(1)} - \mathbf{r}_{23}^{(1)}) + (\mu_0 + \mu_1)(\mathbf{r}_{33}^{(1)} - \mathbf{r}_{32}^{(1)}).
\end{aligned} \tag{10}$$

Let us assume that the patch $\mathbf{r}^{(1)}(u,v)$ is completely defined. All vertices $\mathbf{r}_{00}^{(1)}$... $\mathbf{r}_{33}^{(1)}$ of the characteristic polyhedron for this patch are fixed (Fig. 12, a). We should add the patch $\mathbf{r}^{(2)}(u,v)$ smoothly connected with the fixed patch $\mathbf{r}^{(1)}(u,v)$. The coordinates of the vertices $\mathbf{r}_{10}^{(2)}$, $\mathbf{r}_{11}^{(2)}$, $\mathbf{r}_{12}^{(2)}$, $\mathbf{r}_{13}^{(2)}$ of the patch $\mathbf{r}^{(2)}$ are calculated based on conditions (10). The vertices $\mathbf{r}_{00}^{(2)}$, $\mathbf{r}_{01}^{(2)}$, $\mathbf{r}_{02}^{(2)}$, $\mathbf{r}_{03}^{(2)}$ defining the seam line coincide with the corresponding vertices of the patch $\mathbf{r}^{(1)}$. The remaining eight vertices of the characteristic polyhedron of the patch $\mathbf{r}^{(2)}$ can be defined arbitrarily.



a) b)

Fig. 12. Characteristic polyhedra of the Bézier ribbon: a – smooth connection of patches (general case); b – smooth connection of patches (private case)

The simplest solution will be obtained if we put $\mu_0=\mu_1=0$ in (10). In this case, according to (10), the edges of the characteristic polyhedra of adjacent patches converging on the seam line should be collinear. The ratio of the lengths of these edges should be constant and equal to λ. The numerical value of this parameter is assigned based on design considerations. For example, Fig. 12, b shows the characteristic polyhedra of adjacent Bézier patches at $\lambda=1$, $\mu_0=\mu_1=0$. In this case, the longitudinal generatrices of the patches are smoothly (without a break) connected on the seam line. This method of connecting Bézier patches can be used if the composite boundaries between *ABM* and *DCN* are smooth (see Fig. 11).

This method can be used to connect Bézier patches of different shapes. For example, Fig. 13 shows a computer model of a fairing containing a triangular Bézier patch.

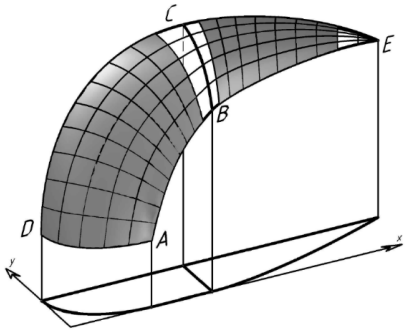
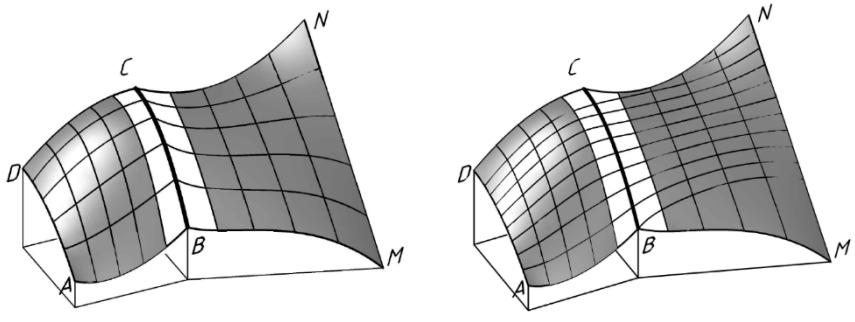


Fig. 13. Fairing

If the coefficients μ_0 and μ_1 are non-zero, we obtain a smooth composite surface, the longitudinal boundaries *ABM* and *DCN* of which are broken at the connection points *B* and *C*. In this case, the requirement of collinearity of edges converging on the seam line is replaced by a weaker requirement of coplanarity. According to the first equation from (10), the edges $(\mathbf{r}_{20}^{(1)}-\mathbf{r}_{30}^{(1)})$, $(\mathbf{r}_{00}^{(2)}-\mathbf{r}_{10}^{(2)})$, $(\mathbf{r}_{00}^{(2)}-\mathbf{r}_{01}^{(2)})$ converging at the point $\mathbf{r}_{30}^{(1)}=\mathbf{r}_{00}^{(2)}$ on the seam line should be coplanar. According to the fourth equation from (10), the three edges converging at the point $\mathbf{r}_{33}^{(1)}=\mathbf{r}_{03}^{(2)}$ should be also coplanar (see Fig. 12, a). The second and third equations from (10) allow calculating the coordinates of the vertices $\mathbf{r}_{11}^{(2)}$ and $\mathbf{r}_{12}^{(2)}$.

The fulfillment of conditions (10) ensures the smoothness of the composite surface for any values of the coefficients λ, μ_0, μ_1. In this case, the longitudinal generatrices break on the seam line without violating the smoothness of the surface, since at the break point on the seam line, there are three coplanar tangents to the composite surface (the tangent to the seam line, the tangent to the "left" longitudinal generatrix, and the tangent to the "right" longitudinal generatrix), forming a single tangent plane. The same applies to corner points *B* and *C*. For example, Fig. 14, a shows a surface calculated for $\lambda = 1$; $\mu_0=-0{,}5$; $\mu_1=-1$. This surface appears to have a break on the seam line *BC*. However, the vertical cutting planes intersect the composite surface along smooth curves (Fig. 14, b), which confirms that the composite surface is smooth.



a) b)

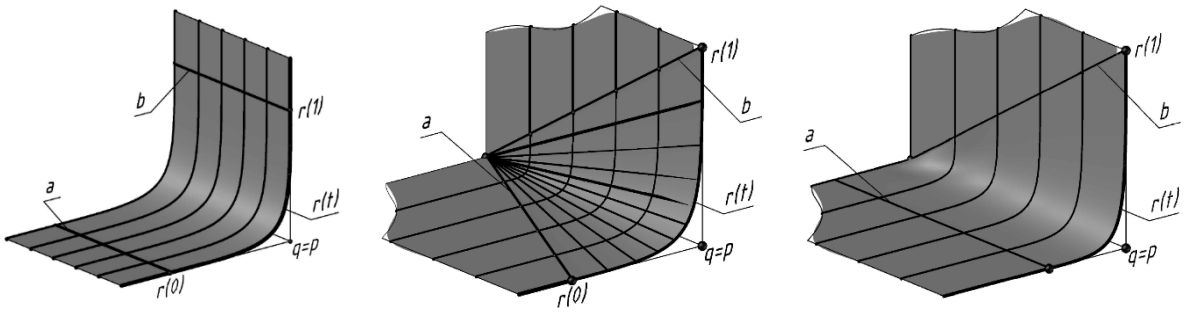
Fig. 14. Smooth Bézier ribbon with broken generatrices: a – generatrices (Bézier curves) with a break on the seam line; b – section of the surface by vertical planes

7. Conjugation of planes

Let us consider the problem of forming a G2-smooth conjugation of two planes. G2 smoothness (continuity) means that the curvature of the composite surface changes smoothly, without jumps in magnitude and direction. Specifically, on the conjugation lines of the plane and the conjugating surface, the curvature of the latter should be zero. This requirement can be satisfied if the transition surface is defined using cubic Bézier curves. According to [4, 5], if the control points **q=p** of a Bézier curve segment coincide

$$\mathbf{r}(t) = (1-t)^3\mathbf{r}(0) + 3t(1-t)^2\mathbf{q} + 3t^2(1-t)\mathbf{p} + t^3\mathbf{r}(1), \tag{11}$$

the curvature of the segment $\mathbf{r}(t)$ at the end points $\mathbf{r}(0)$, $\mathbf{r}(1)$ is equal to zero. If we shift the curve $\mathbf{r}(t)$ in the transverse direction, we obtain a kinematically defined transition surface with zero curvature on the conjugation lines *a*, *b* with the horizontal and vertical planes (see Fig. 15, a).



a) b) c)

Fig. 15. Conjugation of planes with G2 smoothness: a – conjugation lines a, b are parallel; b – con-jugation lines intersect; c – conjugation lines cross

The proposed method of the smooth conjugation of planes can be extended to arbitrarily oriented conjugation lines *a, b*. These lines intersect in Fig. 15, b and cross in Fig. 15, c. In all cases the required G2-smooth conjugation of planes is achieved.

8. Private case: coincidence of Coons and Bézier surfaces

According to the Coons method [13, p. 179], the patch $\mathbf{r}(u,v)$ of a surface spanned by the frame *ABCD* (Fig. 16, a) is defined as a superposition of three ruled surfaces

$$\mathbf{r}(u,v) = \mathbf{r}_1(u,v) + \mathbf{r}_2(u,v) - \mathbf{r}_3(u,v), \tag{12}$$

where

$$\begin{aligned}
\mathbf{r}_1(u,v) &= \alpha_0(u)\mathbf{r}(0,v) + \alpha_1(u)\mathbf{r}(1,v),\\
\mathbf{r}_2(u,v) &= \alpha_0(v)\mathbf{r}(u,0) + \alpha_1(v)\mathbf{r}(u,1),\\
\mathbf{r}_3(u,v) &= \alpha_0(v)[\alpha_0(u)\mathbf{A} + \alpha_1(u)\mathbf{B}] + \alpha_1(v)[\alpha_0(u)\mathbf{D} + \alpha_1(u)\mathbf{C}].
\end{aligned} \tag{13}$$

The surface $\mathbf{r}_1(u,v)$ passes through the "left" $\mathbf{r}(0,v)$ and "right" $\mathbf{r}(1,v)$ boundaries of the patch. The surface $\mathbf{r}_2(u,v)$ passes through the "lower" $\mathbf{r}(u,0)$ and "upper" $\mathbf{r}(u,1)$ boundaries. The surface $\mathbf{r}_3(u,v)$ is an oblique plane based on the non-planar quadrilateral *ABCD*.

If we need the patch $\mathbf{r}(u,v)$ to pass through the given boundaries, the mixing functions α_0, α_1 should satisfy the following conditions:

$$\alpha_0(0) = 1, \quad \alpha_1(0) = 0, \quad \alpha_0(1) = 0, \quad \alpha_1(1) = 1. \tag{14}$$

The simplest set of mixing functions satisfying conditions (14) has the form

$$\alpha_0(u) = 1-u, \quad \alpha_1(u) = u, \quad \alpha_0(v) = 1-v, \quad \alpha_1(v) = v. \tag{15}$$

The use of linear mixing functions (15) ensures that the Coons patches do not combine into a single G1-smooth surface. This problem can be resolved if cubic mixing functions are used to form the patch (12):

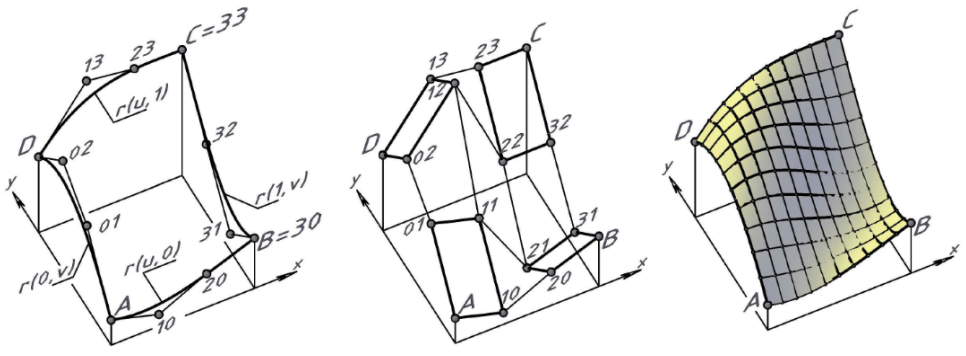
$$\alpha_0(u) = 1 - 3u^2 + 2u^3, \quad \alpha_1(u) = 3u^2 - 2u^3,$$
$$\alpha_0(v) = 1 - 3v^2 + 2v^3, \quad \alpha_1(v) = 3v^2 - 2v^3. \tag{16}$$

Let us formulate the following statement (without proof): if the mixed derivatives $\frac{\partial^2 \mathbf{r}}{\partial u \partial v}$ at the angles of the Bézier patch $\mathbf{r}(u,v)$ spanned by a quadrilateral frame *ABCD* are zero ("plane angles"), the Coons patch (12), (16) spanned by the same frame is identical to the Bézier patch. Notably, the Bézier patch does not contain plane angles in the general case. Unlike the Bézier patch, all angles of the Coons patch (12), (16) are plane.

To formulate the conditions for the existence of plane angles of a Bézier patch, we need to find algebraic expressions for the mixed derivatives at the angles of the patch and equate them to zero. We successively differentiate the Bézier patch equation (2) with respect to the u and v parameters and substitute the sets of values $A\{u=v=0\}$, $B\{u=1,\ v=0\}$, $C\{u=v=1\}$, $D\{u=0,\ v=1\}$ to obtain:

$$\begin{aligned}
\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\Big|_A &= 9[(\mathbf{r}_{00} - \mathbf{r}_{01}) + (\mathbf{r}_{11} - \mathbf{r}_{10})] = 0,\\
\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\Big|_C &= 9[(\mathbf{r}_{22} - \mathbf{r}_{23}) + (\mathbf{r}_{33} - \mathbf{r}_{32})] = 0,\\
\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\Big|_D &= 9[(\mathbf{r}_{12} - \mathbf{r}_{02}) + (\mathbf{r}_{13} - \mathbf{r}_{03})] = 0,\\
\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\Big|_B &= 9[(\mathbf{r}_{20} - \mathbf{r}_{21}) + (\mathbf{r}_{31} - \mathbf{r}_{30})] = 0.
\end{aligned} \tag{17}$$

Geometrically, conditions (17) mean that vertices 11, 12, 21, and 22 of the characteristic polyhedron form parallelograms at the angles of the Bézier patch (Fig. 16, b). In this case, the Bézier patch acquires plane angles, and the Bézier and Coons patches spanned by the *ABCD* frame coincide identically (with an accuracy equal to the accuracy of the electronic model) (Fig. 16, c).



a) b) c)

Fig. 16. Patch of a Bézier surface with plane angles: a – patch boundaries; b – characteristic polyhedron; c – coincidence of the Coons and Bézier patches

9. Software implementation

Any software product that supports three-dimensional graphics and allows drawing spatial Bézier curves specified by their characteristic polyline can be used to construct a Bézier surface. In this paper, all constructions were performed in the AutoCAD environment using the AutoLISP programming language. Due to the lack of licensing, the examples discussed in this paper are not transparent. Unfortunately, the authors were unable to find a similar Russian software product for interactive spatial construction.

According to the proposed algorithms, a grid of longitudinal and transverse generatrices should be drawn, followed by visualization using the "loft" command to obtain a computer model of a surface. Each generatrix is formed as a one-parameter set of points with any, arbitrarily small, step based on the u or v parameter. The coordinates of the points of the generatrix are calculated by expanding its vector equation along the x, y, z coordinate axes. These points are connected using the "spline" command.

10. Conclusion

1. This paper considers various options for modeling complex-shaped surfaces using cubic Bézier curves as generatrices. The generatrix construction algorithm described in Section 2 hereof allows defining the Bézier bicubic patch spanned by a quadrangular or triangular closed contour generated by cubic Bézier curves.

2. The paper considered geometric conditions, the fulfillment of which allows obtaining a Bézier patch with plane generatrices lying in vertical planes (see Section 4).

3. The smoothness conditions proposed by P. Bézier in 1972 are used to obtain a smooth (without folds and breaks) composite Bézier surface. Under these conditions, the generatrix grid can be broken at the seam line, while the composite surface remains G1-smooth (see Section 6).

4. Some properties of the Bézier curve allow forming a G2-smooth conjugation of two planes (see Section 7).

5. It is shown that, under certain boundary conditions, the Coons and Bézier methods yield identical surfaces, which indicates a fundamental commonality between these methods (see section 8).

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