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Multiscale Analysis of High Resolution Digital Elevation Models Using the Wavelet Transform

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Abstract

A technique is proposed for choosing the optimal wavelet basis in terms of decorrelation of the spectral coefficients of the wavelet basis when solving the problem of representation of digital elevation models. In the course of the work, it was revealed that the selection of the spectral transform basis significantly affects the accuracy of the representation of the original model. The proposed method to the decomposition of digital elevation models based on the discrete wavelet transform does not require large computational costs. A technique is proposed for selection the optimal wavelet basis from the position of the minimum mean square error of the reconstructed signal, when quantizing the high-frequency expansion coefficients. Expressions are obtained for generating scaling and wavelet functions in space. The method developed to represent digital elevation models has good properties, which allows to significantly increase the resolution of digital elevation models in the implemented regional geoinformation system.

Keywords: digital elevation models, data visualization, wavelets, multiresolution, discrete wavelet transform, digital twins, terrain model.

1. Introduction

In recent years, there has been a transition in the Russian Federation to the provision of public services from the traditional form to the electronic one. In the vast majority of countries of the world, government initiatives are the main engine of the development of informatization. [1]. In this regard, the Ministry of Telecom and Mass Communications of the Russian Federation is conducting systematic work aimed at improving the quality and level of accessibility of state and municipal services in electronic form.

One of the requirements involves the possibility of operational work with a large amount of various data archives, including the search for the desired element [2]. In the case of complex geographically distributed systems for the implementation of mobile management and related analysis, it becomes necessary to apply and develop cartographic services, the functionality of which supports customized layers for each industry or consumer group. Otherwise, potential consumers of the service will be disoriented, which in turn generates lost profits for all market participants.

In recent years, cartographic services and geographic information systems have gradually moved from using traditional two-dimensional maps to three-dimensional visualization.

With the development of satellite communications and the advent of open services for accessing globally consistent digital elevation data with high spatial resolution, such as ASTER Global Digital Elevation Map, Earth remote sensing data have gained the most popularity.

All multiple regional information services, most of which are built as cartographic services, should preferably be based on the concept of a digital twin of the region's territory, the use of which allows you to reduce financial and time costs.

Digital twins in the management of the region are promising systems that allow you to create an interface between cities and infrastructure and various information systems, make the necessary variable changes and analyze their effectiveness.

For example, the scheme of territorial planning of capital construction of gas supply facilities in the Volgograd region is shown in Figure 1. It should be noted that this is far from the only working urban web service.

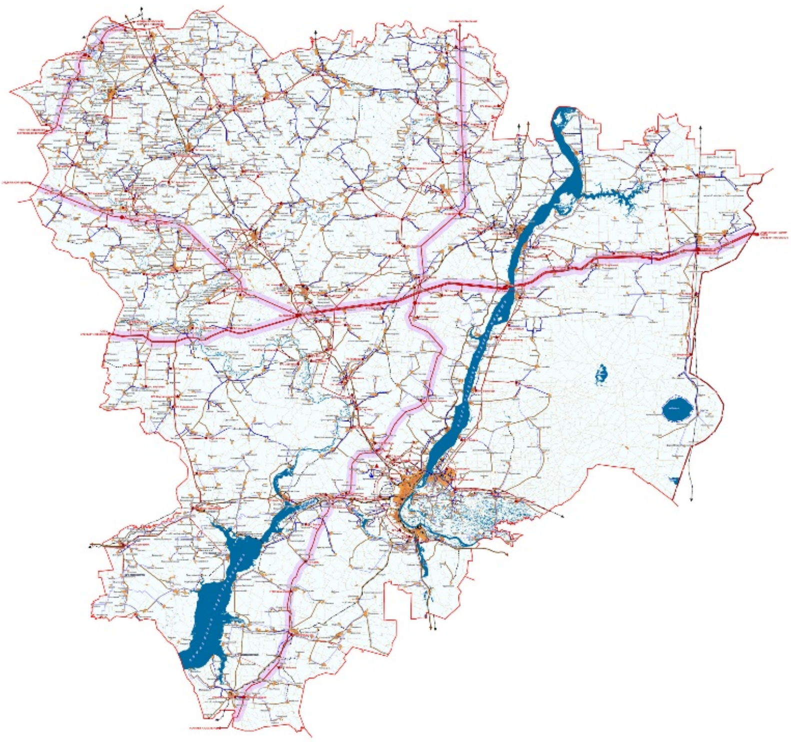

Fig. 1. The scheme of the territorial planning of capital construction of gas supply facilities in the Volgograd region.

The above scheme of territorial planning of gas supply facilities is narrow-profile but contains a large number of existing and under construction objects of various types, such as gas pipelines, gas distribution stations, inter-settlement gas pipelines, underground gas storages. All these objects have many parameters to describe them in databases.

The main problem in the design and development of cartographic services based on the use of a digital twin of the territory of the region is the availability of technologies for their implementation, including high-resolution visualization of the relief of the Earth's surface [3].

One of the tasks of visualizing digital elevation data is to create effective methods for coding them for subsequent visualization for its full interpretation. In conditions of extreme redundancy by nature of high resolution digital elevation data, this problem is one of the most urgent, and its solution directly affects the computational efficiency and, consequently, the achievement of the Service Level Agreements.

The key role in the process of interpreting and analyzing data is played by the visualization of the elements of the digital twin of the territory of the region, this is a technique for converting analytical abstraction about objects into geometric views using computer graphics core as part of the visualization software.

Digital elevation data visualization problem is shown in Figure 2. Digital elevation data is characterized by redundant data representation. For example, an only array of 32-bit digital elevation data elements with a size of 10x10 km with a detail of 1 m will occupy 3200 MB.

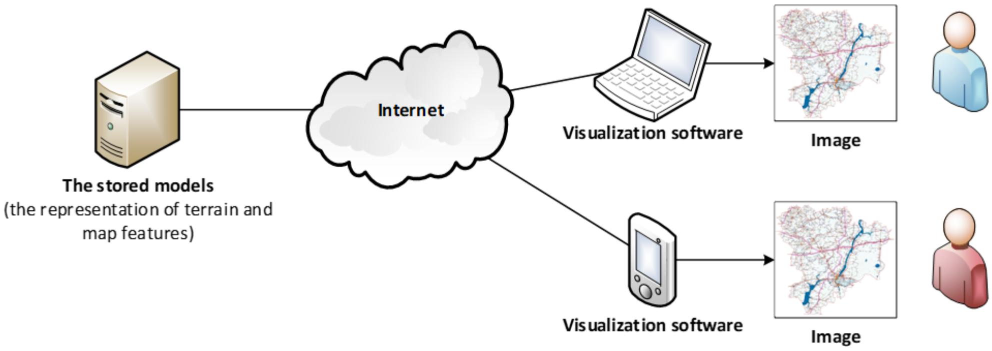

Fig. 2. Digital elevation data visualization process.

As a consequence, the transmission of digital elevation data via communication channels with limited bandwidth [4], and subsequent loading into the RAM of a personal computer or smartphone is expensive, and therefore the task of compact representation of the relief of the Earth's surface is especially relevant.

2. Digital Elevation Models

A typical two-dimensional map, for example, generated during route planning by navigation software, does not contain additional information about the terrain, such as passing one road over another without crossing them, leading to errors for drivers trying to navigate such a map.

A modern alternative to such a display is the development of a digital twin of the territory of the region [5], which is capable of displaying the environment familiar to a person in the form of a three-dimensional image with objects proportional to each other: buildings, shops, bus stops, green spaces, car washes, gas stations, pharmacies, restaurants, hairdressers, etc.

In addition, three-dimensional maps allow you to orientate the sky around the visual features of the area: hills, forests, fields, mountainous terrain on the horizon.

When solving such problems as the territorial planning of capital construction of gas supply facilities in the region, or the modernization of the mobile communication network, it is also necessary to take into account the features of the terrain.

A digital elevation model is a digital model or digital cartographic dataset that has a three-dimensional representation of the terrain surface, created from elevation data [6].

There are several methods that can be used to generate digital elevation models: ground surveying, Light Detection and Ranging (LiDAR), radar interferometry, multibeam echo sounder [7].

Digital elevation models are a core spatial dataset required for many environmental, planning, and scientific applications: urban and environmental planning, topographic maps, emergency responses, hydrological functions, geological survey and analyses, transportation planning, military applications, maritime transportation, etc.

3. Discrete wavelet transform

In the field of representation of signals and images, presentation methods are distinguished that are focused on stationary and non-stationary signals [8].

The idea of representing multidimensional signals based on wavelet transforms, as well as other methods based on orthogonal transforms, is quite simple. Initially, as a result of the transform, some spectral components are removed from the resulting data set according to the specified criteria. The remaining set of coefficients is usually encoded. The dataset is recovered by decoding the coefficients, if necessary, and applying an inverse transform to the result [9]. It is assumed that not too much information is lost during the decimation of the transform coefficients.

However, it should be noted that, in order to obtain a high value of accuracy and compression ratio, it is necessary to choose the optimal wavelet basis for this type of signals.

Currently, it has become obvious that the Fourier transform, which is a traditional approach for analyzing stationary signals, is ineffective for representing signals with local features that have been widely used in recent years.

One of the main disadvantages of the Fourier transform is the use of a sine wave as a basis, because it has an infinite scope of definition. In the case of a limited number of terms of the Fourier series, a basis of this type is fundamentally incapable to represent non-stationary signals [10].

The set of all measurable f defined on the interval $(0, 2\pi)$ denote by $L^2(0, 2\pi)$, then we have:

$$\int_0^{2\pi} |f(x)|^2 dx < \infty. \quad (1)$$

The domain of the piecewise continuous function is $D(f) = R$, so $L^2(0, 2\pi)$ is a 2π-periodic function with root mean square convergence. Any function f can be represented as a Fourier series [11]:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad (2)$$

where c_n are the Fourier coefficients, which are defined as:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \quad (3)$$

The meaning of the Fourier series expansion is that some function $f \in L^2(0, 2\pi)$ п is represented as a set of integer extensions of the basis function $\omega(x) = e^{ix} = cos(x) + i\, sin(x)$. The function $\omega_n(x) = \omega(nx)$ which is a sine wave does not belong to the space $L^2(R)$ for which the expression is valid: $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. Accordingly, any function $f \in L^2(R)$ must decay at $x \to \pm\infty$ and $\omega_n(x) \notin L^2(R)$, and to generate the space $L^2(R)$, which is one of the cases of a Hilbert space, it is necessary that $\omega_n(x)$ decay to zero as $x \to \pm\infty$ as, for example, in case of Le Gall basis functions[12].

Wavelet analysis also allows any function f can be represented in terms of the basis of functions ψ_i:

$$f = \sum_i c_i \psi_i. \quad (4)$$

The wavelet transform of function f is a sum of the basis functions ψ_i, weighted by the coefficients c_i [13, 14]. The basis functions ψ_i are given, and only the coefficients c_i contain information about the original function f. The number of wavelet coefficients c_i is a criterion of efficiency. A more efficient wavelet transform allows to represent the original function f with fewer wavelet coefficients. This is possible if the wavelet function ψ_i localizes the features of the original function f more precisely, but real signals are localized in both the time and frequency domains [15] so the wavelet transform is better than Fourier transform because the original function f is decomposed into different frequency bands.

4. Principles of multiresolution decomposition

The concept of multiresolution decomposition is based on the representation of a signal as a combination of its rough representation and detailed local representations of the signal, i.e. a sequence $\{V_j\}_{j\in Z}$ of closed subspaces $L^2(R)$ satisfying the following properties [16]:

$$V_j \subset V_{j+1}; \quad (5)$$

$$\overline{\cup_{j\in Z} V_j} = L^2(R); \quad (6)$$

$$\cap_{j\in Z} V_j = \{0\}; \quad (7)$$

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}; \quad (8)$$

$$f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0. \quad (9)$$

A Riesz basis is a Riesz sequence that is a basis in the Hilbert space H [17]. According to this definition, it can be argued that there exists a function $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in Z}$ forms a Riesz basis in V_0.

The multiresolution decomposition describes a sequence of nested approximation spaces V_j in $L^2(R)$, so all subspaces V_j are uniquely determined from the space V_0 by changing the scale of the function ϕ. It is called scaling function if it is possible to perform a multiresolution decomposition [18] . 2 is selected as the scaling factor, because it is based on a computationally fast shift operation, i.e. the function f of the space V_j when scaled by 2, becomes an element of the space V_{j+1} with orthonormal bases: $\langle e_{n,}e_m\rangle = \delta_{nm}$, and $\|f\|^2 = \int|f(x)|^2dx = \sum_n|\langle f, e_n\rangle|^2$.

Let's put $\phi_{j,k}(x) = 2^{j/2}\phi(2^jx - k); j, k \in Z$, then it follows from (8) and (9) that the sequence $\{\phi_{j,k}\}_{k\in Z}$ is a Riesz basis in V_j for any $j \in Z$, then there is a sequence $\{h_k\}_{k\in Z}$ for scaling function:

$$\phi(x) = \sqrt{2}\sum_{k\in Z} h_k\phi(2x - k). \tag{10}$$

The space W_j is the orthogonal complement of V_j в V_{j+1}, i.e. $V_{j+1} = V_j \oplus W_j$ and $W_j \perp V_j, j \in Z$. Due to (6) and (7) there is a representation of the multiresolution representation with decomposition of the space V_{j+1} into the space V_j and orthogonal addition W_j:

$$L^2(R) = \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots = \bigoplus_{j\in Z} W_j. \tag{11}$$

A wavelet function is a function $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k), j, k \in Z$, that the sequence $\{\psi_{j,k}\}_{k\in Z}$, as a consequence of the definition of the scaling function $\phi(x)$ from (10), is a Riesz basis in W_j for any $j \in Z$, then there is a sequence of scaling ratios $\{g_k = (-1)^k h_{1-k}\}_{k\in Z}$:

$$\psi(x) = \sqrt{2}\sum_{k\in Z} g_k\phi(2x - k), \tag{12}$$

where $\{h_k\}$ are the filter coefficients for the scaling function $\phi(x)$, $\{g_k\}$ are the filter coefficients for the wavelet function $\psi(x)$. The filter coefficients $\{h_k\}$ and $\{g_k\}$ make it possible to calculate the values of the functions ϕ and ψ, as well as the spectral coefficients of the discrete wavelet transform $\{d_j\}$.

By virtue of the relation $V_j = V_{j-1} \oplus W_{j-1}$ it seems possible to define the function $f(x)$, written in terms of the basis functions of the space V_j, in terms of the basis functions of the spaces V_{j-1} and W_{j-1} taking into account the expressions for the scaling function $\phi(x)$ and the wavelet function $\psi(x)$, then the discrete wavelet transform is defined as some mapping W, that translates the sequences $\{c_{j,k}\}$ in sequences $\{c_{j-1,k}, d_{j-1,k}\}$.

The wavelet transform decompose the signal in some system of functions that are shifted and scaled copies of the basis function, then any function $f \in L^2(R)$ can be represented on some given resolution level j_n as:

$$f(x) = \sum_k c_{j_n,k}\phi_{j_n,k}(x) + \sum_{j_n}\sum_k d_{j,k}\psi_{j,k}(x), \tag{13}$$

where the first term of the sum is a rough approximation of the original function f, which is an element of the low-frequency bands of the spectrum. As a result of the successive addition of other detail terms of the series to the rough approximation of the original function f, the resolution of the reconstructed function increases.

5. Approximation of Digital Elevation Models

To represent multidimensional digital elevation model, the multi-scale decomposition must be performed in the R^3 space or higher order. There are two main approaches that allow the approximation of multidimensional digital elevation data.

The first approach is based on the computation of a wavelet transform using multidimensional wavelet basis. Unfortunately, it has a high computational complexity and is therefore rarely used.

The second approach is to combine one-dimensional wavelet transforms. In this case, the functions of this wavelet transform are tensor products of the functions of a one-dimensional wavelet transform.

For example, for a wavelet transform in R^2, four generating functions must be specified:

$$\begin{aligned} \Phi(x,y) &= \phi(x)\phi(y), \\ \psi^{LH}(x,y) &= \phi(x)\psi(y), \\ \psi^{HL}(x,y) &= \psi(x)\phi(y), \\ \psi^{HH}(x,y) &= \psi(x)\psi(y). \end{aligned} \tag{14}$$

The remaining functions of the wavelet transform are determined by the relation:

$$\begin{gathered}\Phi_{k,m}(x,y) = 2^j\Phi(2^jx-k,2^jy-m),\\ \Psi_{k,m}^{LH}(x,y) = 2^j\Psi^{LH}(2^jx-k,2^jy-m),\\ \Psi_{k,m}^{HL}(x,y) = 2^j\Psi^{HL}(2^jx-k,2^jy-m),\\ \Psi_{k,m}^{HH}(x,y) = 2^j\Psi^{HH}(2^jx-k,2^jy-m).\end{gathered} \tag{15}$$

The space V_j generated by shifts of the scaling function $\Phi(x,y)$ on the same scale j, and the detail spaces have the form:

$$\begin{gathered}V_{k.m} = \{2^j\Phi(2^jx-k,2^jy-m)\},\\ W_{k.m}^{LH} = \{2^j\Psi^{LH}(2^jx-k,2^jy-m)\},\\ W_{k.m}^{HL} = \{2^j\Psi^{HL}(2^jx-k,2^jy-m)\},\\ W_{k.m}^{HH} = \{2^j\Psi^{HH}(2^jx-k,2^jy-m)\}.\end{gathered} \tag{16}$$

For the wavelet transform in R^3 we define eight generating functions:

$$\begin{gathered}\Phi(x,y,z) = \phi(x)\phi(y)\phi(z),\\ \psi^{LLH}(x,y,z) = \phi(x)\phi(y)\psi(z),\\ \psi^{LHL}(x,y,z) = \phi(x)\psi(y)\phi(z),\\ \psi^{LHH}(x,y,z) = \phi(x)\psi(y)\psi(z),\\ \psi^{HLL}(x,y,z) = \psi(x)\phi(y)\phi(z),\\ \psi^{HLH}(x,y,z) = \psi(x)\phi(y)\psi(z),\\ \psi^{HHL}(x,y,z) = \psi(x)\psi(y)\phi(z),\\ \psi^{HHH}(x,y,z) = \psi(x)\psi(y)\psi(z)\end{gathered} \tag{17}$$

Then the remaining functions of the wavelet transform can be described by the relation:

$$\begin{gathered}\Phi_{k,m,l}(x,y,z) = 2^{3j/2}\Phi(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{LLH}(x,y,z) = 2^{3j/2}\Psi^{LLH}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{LHL}(x,y,z) = 2^{3j/2}\Psi^{LHL}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{LHH}(x,y,z) = 2^{3j/2}\Psi^{LHH}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{HLL}(x,y,z) = 2^{3j/2}\Psi^{HLL}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{HLH}(x,y,z) = 2^{3j/2}\Psi^{HLH}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{HHL}(x,y,z) = 2^{3j/2}\Psi^{HHL}(2^jx-k,2^jy-m,2^jz-l),\\ \Psi_{k,m,l}^{HHH}(x,y,z) = 2^{3j/2}\Psi^{HHH}(2^jx-k,2^jy-m,2^jz-l).\end{gathered} \tag{18}$$

The space V_j generated by shifts of the scaling function $\Phi(x,y,z)$ on the same scale j, and the detail spaces will have the form:

$$\begin{gathered}V_{k,m,l}(x,y,z) = \{2^{3j/2}\Phi(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{LLH}(x,y,z) = \{2^{3j/2}\Psi^{LLH}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{LHL}(x,y,z) = \{2^{3j/2}\Psi^{LHL}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{LHH}(x,y,z) = \{2^{3j/2}\Psi^{LHH}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{HLL}(x,y,z) = \{2^{3j/2}\Psi^{HLL}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{HLH}(x,y,z) = \{2^{3j/2}\Psi^{HLH}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{HHL}(x,y,z) = \{2^{3j/2}\Psi^{HHL}(2^jx-k,2^jy-m,2^jz-l)\},\\ W_{k,m,l}^{HHH}(x,y,z) = \{2^{3j/2}\Psi^{HHH}(2^jx-k,2^jy-m,2^jz-l)\}.\end{gathered} \tag{19}$$

6. Selection of Wavelet Basis Function

When decomposing the digital elevation model, it is important to choose the optimal basis in terms of decorrelation of wavelet coefficients. This problem arises due to the fact that the samples of the original digital elevation model are related to each other with a correlation coefficient reaching values close to one. This fact means that any count can be reconstructed from near samples due to the fact that there is redundancy in digital elevation data.

As an objective criterion for estimating the difference between the original digital elevation data $f(x, y, z)$ and the reconstructed i-th basis for decomposition of digital elevation data $f_i'(x, y, z)$ we derive a criterion based on the mean square error vector for the digital elevation model with dimensions $N \times M \times K$.

Then the problem of choosing the optimal wavelet basis for representing the digital elevation model for a specific given compression ratio will be written as:

$$\frac{1}{\mathrm{N} \times \mathrm{M} \times \mathrm{K}} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \sum_{z=0}^{K-1} \left(f(x, y, z) - f_i'(x, y, z)\right)^2 \rightarrow \min_{i \in S} \quad (20)$$

where S – number of wavelet bases.

7. The numerical experiments

Based on the proposed approach, a library of digital elevation models coding methods was developed. The proposed approach was tested on 56 digital elevation models, which have different morphological and spectral characteristics. To approximate the initial models of the 3-dimensional surface relief, 89 different wavelet bases were used.

As an optimality criterion, the minimum mean square error of the reconstructed digital elevation model obtained as a result of compression from the original model is used at a given constant compression ratio. Since this criterion is not always consistent with the visual perception of the quality of the resulting image, it is possible to visually check the results obtained. An additional analysis of the digital elevation models spectrum obtained as a result of the wavelet transform allows us to evaluate how effectively this transform localizes energy in the low-frequency region.

An example of approximation of the original model of the 3-dimensional surface relief using the Le Gall basis [19] is shown in Figure 3.

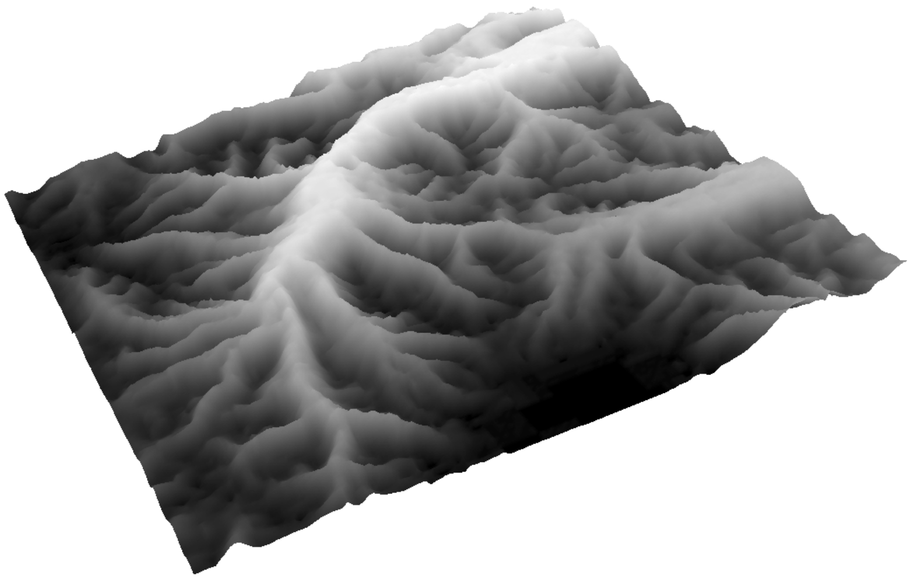

Fig. 3. Approximation of the original model of the 3-dimensional surface relief with the Le Gall basis, compression ratio is 16:1.

Figure 4 shows an estimate of the proportion of distortions during compression of one of the 3-dimensional surface relief models, presented using various wavelet bases.

It can be seen that the approximations of the original digital elevation models based on the Antonini and Brislawn wavelets have higher quality indicators than the approximations of the original elevation model based on the traditional Haar, Daubechies db2, or Villasenor 1810 transformations under the same conditions, which is explained by their lower efficiency in representing original digital elevation models.

The values of the optimality of representation and the proportion of introduced distortions using different wavelet transforms for the same digital elevation models can differ significantly, which allows us to conclude that it is important to choose the optimal wavelet basis and the efficiency of representing digital elevation models by the selected wavelet basis.

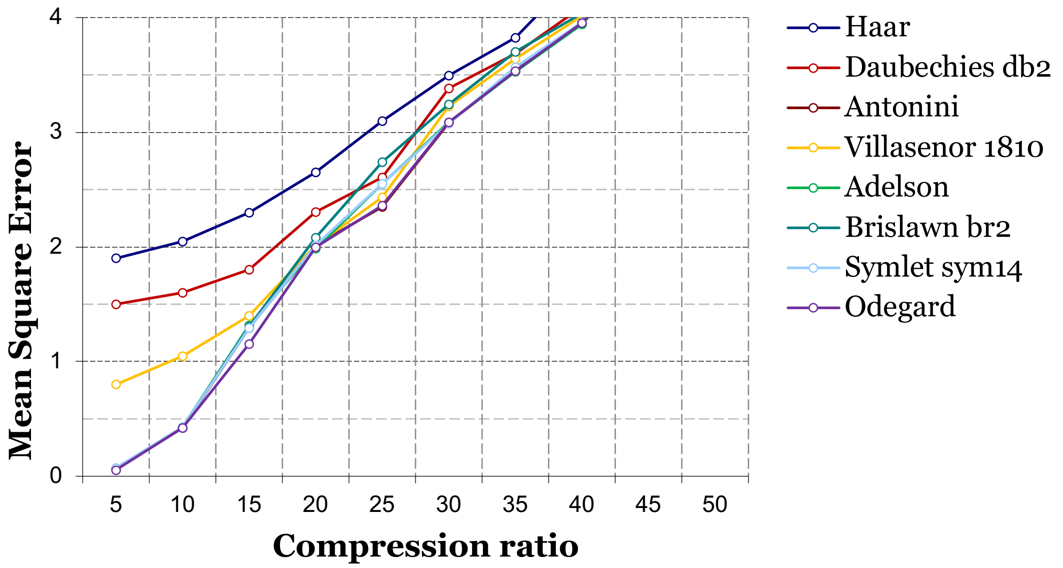

Fig. 4. Dependence of the root mean square error on the compression ratio for various wavelet bases.

The selection of the optimal wavelet basis for decomposition of digital elevation model in the general case is a complex and difficult to formalize problem. There are many properties known for constructing an optimal wavelet basis, among which the most important are: orthogonality, smoothness, orthonormality, approximation accuracy, normalization, symmetry, size of the definition area. However, the optimal combination of these parameters for a specific digital elevation model is unknown.

In addition, a basis that works well for one digital elevation models class may not work at all for another class. To select the optimal wavelet basis, modeling of the quantization process is used, since the analytical solution of this problem is rather complicated and up to now there is no mathematical model that effectively describes all types of triangulation models.

Conclusions

A technique is proposed for choosing the optimal wavelet basis in terms of decorrelation of the wavelet coefficients when solving the problem of representing digital elevation models. It is shown that the efficiency of the selection of basis significantly affects the error in computing the inverse spectral transform to reconstruction of digital elevation models. As a result of the analysis, it was found that stationary signals are effectively represented using classical decomposition methods, and non-stationary signals, which include digital elevation models, are more efficiently represented using wavelet transforms with an optimal basis. A technique for solving the problem of representation for subsequent processing of digital elevation models, adapted to its main features, is given.

The proposed method for selection of optimal wavelet basis for representation of digital elevation model based on the wavelet transform has linear computational complexity.

The important features of the proposed method include the use of a single approach for representing and processing digital elevation models, which made it possible to increase the number of possible operations, for example, to process a fragment of the model at a given scale. It was possible to achieve high accuracy in the representation of digital elevation models, as well as to reduce the computation time, which significantly increased the quality of service when providing digital elevation models in the implemented regional geoinformation system.

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