Scientific Visualization, 2023, volume 15, number 2, pages 45 - 65, DOI: 10.26583/sv.15.2.05

Bicubic Surface on a Fixed Frame: Calculation and Visualization

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Abstract

The paper proposes an algorithm for calculating a composite bicubic surface with a fixed frame formed by longitudinal (along the x axis) and transverse (along the y axis) cubic splines. Frame line equations are taken as the main boundary conditions. According to the proposed algorithm, the problem is divided into two stages: first, the frame line equations are found and then the coefficients included in the equations of the bicubic portions forming the bicubic surface are calculated. This approach reduces the size of the characteristic matrix of the linear equation system by reducing the number of coefficients in the surface equation. The matrix size is reduced from 16mn to 9mn, where m and n are the number of bicubic portions along the x and y axes. Surface visualization is reduced to building a grid of longitudinal and transverse generators, the equations of which are formed from the surface equation by substituting y=const (longitudinal generators) or x=const (transverse generators).

In this paper we calculate and visualize bicubic surfaces with a frame formed by a mixed set of cubic splines and straight lines. The clarity of the examples is ensured by indicating the numerical values of all calculated magnitudes with an accuracy of up to nine significant figures.

Keywords: bicubic portion, cubic spline, curvature, smoothness, gradient, plane angles, fixed end points.

1. Introduction

A bicubic surface is formed by bicubic portions

$$\Phi(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{ij}x^3y^3\ , \qquad (1)$$

bounded by the cells of a rectangular grid on the *xy* plane of the Cartesian coordinate system *Oxyz*. Portions (1) are interconnected and have a preset degree of smoothness.

The equation of each portion contains 16 coefficients, which are determined based on the incidence, smoothness, and boundary conditions. The surface on the $m\times n$ grid consists of *mn* bicubic portions of form (1), each of which is determined by its own set of 16 coefficients a_{ij}. The calculation of the coefficients is reduced to solving the system of linear algebraic equations with a 16*mn* square characteristic matrix. For example, a system of 64 linear equations must be solved to calculate the coefficients of the equation of a surface formed by four bicubic portions.

Scientific novelty. The paper proposes an algorithm which can nearly halve (from 16*mn* to 9*mn*) the size of the characteristic matrix. As opposed to known algorithms, the constructed surface is considered as a set of longitudinal (elongated along the *x*-axis) bicubic tapes connected in the transverse direction with C^2 smoothness (with continuous changes in curvature when crossing the common boundary of the two tapes). Each tape is formed from sequentially connected bicubic portions. We formulated and proved the algebraic conditions of C^2-smooth bicubic tape connection.

Practical significance. Composite surfaces are often modeled in modern architecture when designing structures with sufficient spatial freedom. In particular, the search for new non-linear forms has led to the appearance of "tent architecture" [1] and "fold architecture" [2], which use overlaps with complex curvilinear outlines. If a constructed surface has no large gradients relative to a base plane *xy*, bicubic polynomials in the scalar values *x, y* can be effectively used to model it [3, 4, 5].

2. Problem statement

A rectangular grid $(x_0, x_1, ..., x_m)\times(y_0, y_1, ..., y_n)$ is marked on the *xy* plane. Points with different elevations are indicated in the nodes of this grid. The angular and boundary points have gradients (the slope angles of the constructed surface to the *xy* plane) in the longitudinal (along the *x* axis) and transverse (along the axis) directions. It is required to form a rectangular C^2-smooth surface with given gradients passing through the specified points. The C^2 smoothness means a continuous (without "jumps") change in the surface curvature at any point and in any direction [6, 7].

We will construct a surface from bicubic portions of form (1) bounded by the cells of the rectangular grid $m\times n$, where *m, n* are positive integers. At $m=n=1$, we obtain a bicubic portion. At $m\geq 2$ and $n=1$, we obtain a bicubic band. At $m=n=2$, we obtain the simplest compound bicubic surface.

Note. A bicubic band is a rectangular-plan surface elongated along the *x* axis, formed by a set of bicubic portions interconnected by transverse joints (with C^2 smoothness).

3. Bicubic portion

A rectangular cell is marked on the *xy* plane of the Cartesian coordinate system *xyz*, where $h_x=x_1-x_0$, $h_y=y_1-y_0$ (Fig. 1).

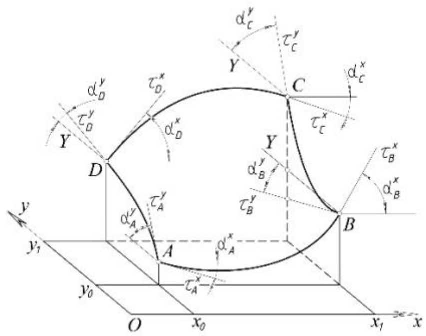


Fig. 1. Boundary conditions

The angular points $A(x_0, y_0, z_A)$, $B(x_1, y_0, z_B)$, $C(x_1, y_1, z_C)$, $D(x_0, y_1, z_D)$ are specified and the equations of boundary curves (cubic parabolas) are given:

$$\begin{aligned} &AB = z_{AB}(x) = \alpha_{AB} + \beta_{AB}(x-x_0) + \gamma_{AB}(x-x_0)^2 + \delta_{AB}(x-x_0)^3, \quad x\in[x_0, x_1], \quad y=y_0, \\ &AD = z_{AD}(y) = \alpha_{AD} + \beta_{AD}(y-y_0) + \gamma_{AD}(y-y_0)^2 + \delta_{AD}(y-y_0)^3, \quad y\in[y_0, y_1], \quad x=x_0, \\ &BC = z_{BC}(y) = \alpha_{BC} + \beta_{BC}(y-y_0) + \gamma_{BC}(y-y_0)^2 + \delta_{BC}(y-y_0)^3, \quad y\in[y_0, y_1], \quad x=x_1, \\ &DC = z_{DC}(x) = \alpha_{DC} + \beta_{DC}(x-x_0) + \gamma_{DC}(x-x_0)^2 + \delta_{DC}(x-x_0)^3, \quad x\in[x_0, x_1], \quad y=y_1. \end{aligned} \tag{2}$$

The free term of the cubic parabola equation is hereinafter denoted by letter α, and the coefficients at the increasing degrees of the argument are denoted by letters β, γ, δ, respectively. The subscripts indicate the boundary points of the relevant parabolas. It is required to find the equation of the bicubic surface (portion) $\Phi(x, y)=ABCD$ "stretched" along the given boundary curves.

The expanded equation of bicubic portion (1) is:

$$\begin{aligned}\Phi(x, y) &= a_{00} + a_{01}(y-y_0) + a_{02}(y-y_0)^2 + a_{03}(y-y_0)^3 + \\ &+[a_{10} + a_{11}(y-y_0) + a_{12}(y-y_0)^2 + a_{13}(y-y_0)^3](x-x_0) + \\ &+[a_{20} + a_{21}(y-y_0) + a_{22}(y-y_0)^2 + a_{23}(y-y_0)^3](x-x_0)^2 + \\ &+[a_{30} + a_{31}(y-y_0) + a_{32}(y-y_0)^2 + a_{33}(y-y_0)^3](x-x_0)^3, \quad x \in [x_0,\ x_1],\ y \in [y_0,\ y_1].\end{aligned} \tag{1a}$$

Assuming that $y=y_0$, we isolate the equation of boundary curve AB from (1a):

$$z_{AB}(x) = a_{00} + a_{10}(x-x_0) + a_{20}(x-x_0)^2 + a_{30}(x-x_0)^3,\ x \in [x_0,\ x_1]. \tag{3}$$

Equating the coefficients included in the first equation from (2) and the coefficients included in equation (3), we obtain:

$$a_{00} = \alpha_{AB}, \quad a_{10} = \beta_{AB}, \quad a_{20} = \gamma_{AB}, \quad a_{30} = \delta_{AB}. \tag{4}$$

Similarly, assuming that $x=x_0$, we isolate the equation of boundary curve AD from (1a) and equate the coefficients at the identical degrees of the variable y:

$$a_{01} = \beta_{AD}, \quad a_{02} = \gamma_{AD}, \quad a_{03} = \delta_{AD}. \tag{5}$$

Assuming that $x=x_1$, we isolate the equation of boundary curve BC from (1a). Assuming that $y=y_1$, we isolate the equation of boundary curve DC from (1a). Equating the coefficients of the obtained equations with the relevant coefficients from (2) and taking into account (4), (5), we obtain the system of five linearly independent equations

$$\begin{aligned}&a_{11}h_x + a_{21}h_x^2 + a_{31}h_x^3 = \beta_{BC} - \beta_{AD}, \\ &a_{12}h_x + a_{22}h_x^2 + a_{32}h_x^3 = \gamma_{BC} - \gamma_{AD}, \\ &a_{13}h_x + a_{23}h_x^2 + a_{33}h_x^3 = \delta_{BC} - \delta_{AD}, \\ &a_{11}h_y + a_{12}h_y^2 + a_{13}h_y^3 = \beta_{DC} - \beta_{AB}, \\ &a_{21}h_y + a_{22}h_y^2 + a_{23}h_y^3 = \gamma_{DC} - \gamma_{AB}\end{aligned} \tag{6}$$

with respect to nine coefficients a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}.

Note. Isolating the equations of the cubic parabolas BC, AD from (1a) and equating the coefficients of these equations to the relevant coefficients from (2) while taking into account (4) and (5), we obtain not five, but six linear equations. We can show that any of these six equations results from the five remaining equations so one equation, namely $a_{31}h_y + a_{32}h_y^2 + a_{33}h_y^3 = \delta_{DC} - \delta_{AB}$, is neglected.

Four boundary conditions should be specified to determine the nine unknown coefficients included in (6). The "plane angles" conditions can be taken as additional boundary conditions: the first mixed derivatives of function (1a) are equal to zero at the angular points of the constructed bicubic portion [8]. Differentiating (1a) and equating the first mixed derivatives to zero, we obtain:

$$\begin{aligned}&\frac{\partial^2\Phi}{\partial x\partial y}(x_0, y_0) = a_{11} = 0, \quad \frac{\partial^2\Phi}{\partial x\partial y}(x_1, y_0) = 2a_{21} + 3h_x a_{31} = 0, \\ &\frac{\partial^2\Phi}{\partial x\partial y}(x_0, y_1) = 2a_{12} + 3h_y a_{13} = 0, \quad \frac{\partial^2\Phi}{\partial x\partial y}(x_1, y_1) = 4a_{22} + 6h_y a_{23} + 6h_x a_{32} + 9h_x h_y a_{33} = 0.\end{aligned} \tag{7}$$

We find the coefficients a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} from the system of equation (6), (7). The remaining seven coefficients of equation (1a) are calculated according to (4), (5). The problem is solved.

Note. Whereas bicubic portion is set by the angular points *A, B, C, D* and the gradients (the slope angles of the tangents τ^x, τ^y at the angular points), equations (2) of the segment boundaries are determined by a simple calculation [9, 10]. For example, if the slope angles α^x_A, α^x_B of the tangents τ^x_A, τ^x_B are set at the finite points of boundary curve *AB* (see Fig. 1), the coefficients γ_{AB}, δ_{AB} included in the equation of this curve are calculated from the system of equations

$$\begin{aligned} &\alpha_{AB}+\beta_{AB}h_x+\gamma_{AB}h_x^2+\delta_{AB}h_x^3=z_B, \\ &\beta_{AB}+2\gamma_{AB}h_x+3\delta_{AB}h_x^2=tg\alpha_B^x, \end{aligned} \tag{8}$$

where $\alpha_{AB}=z_A$, $\beta_{AB}=tg\alpha^x_A$. The equations of other boundaries are determined similarly.

Example 1

We are given the coordinates of the angular points *A*(0; 0; 2,5), *B*(10; 0; 5), *C*(10; 10; 12,5), *D*(0; 10; 7,5). Gradients are fixed at the angular points (see Fig. 1):

$tg\alpha_A^x=-0,4;\quad tg\alpha_A^y=1,7;\quad tg\alpha_B^x=0,5;\quad tg\alpha_B^y=-1;\quad tg\alpha_C^x=0;\quad tg\alpha_C^y=1,7;\quad tg\alpha_D^x=1,7;\quad tg\alpha_D^y=-1.$

The boundary plane angles conditions (the first mixed derivatives are equal to zero at the angles of the portion) are additionally accepted. We must find equation (1a) of the bicubic portion satisfying the conditions of the problem.

Solution

Substituting the values $h_x=h_y=10$ and the coordinates of points *A, B* in (8), we find the coefficients of the equation of boundary curve *AB*. Similarly, calculating the coefficients of the equations of boundary curves *AD, BC* and *DC*, we obtain:

$$\begin{aligned} &z_{AB}(x)=2.5-0.4x+0.105x^2-0.004x^3, \\ &z_{BC}(y)=5-y+0.255y^2-0.008y^3, \\ &z_{AD}(y)=2.5+1.7y-0.09y^2-0.003y^3, \\ &z_{DC}(x)=7.5+1.7x-0.19x^2+0.007x^3. \end{aligned} \tag{9}$$

Substituting the coefficients of equations (9) into (4), (5), we obtain:

$$a_{00}=\alpha_{AB}=2.5;\quad a_{10}=\beta_{AB}=-0.4;\quad a_{20}=\gamma_{AB}=0.105;\quad a_{30}=\delta_{AB}=-0.004;$$
$$a_{01}=\beta_{AD}=1.7;\quad a_{02}=\gamma_{AD}=-0.09;\quad a_{03}=\delta_{AD}=-0.003.$$

The remaining coefficients included in (1a) are found from the system of equations (6), (7):

$$a_{11}=0;\, a_{12}=0.063;\, a_{13}=-0.0042;\, a_{21}=-0.081;\, a_{22}=-0.00075;\, a_{23}=0.00059;$$
$$a_{31}=0.0054;\, a_{32}=-0.00021;\, a_{33}=-0.000022.$$

We determined all the coefficients of equation (1a). Figure 2 shows the grid of generators of the bicubic portion *BCD* constructed according to (1a).

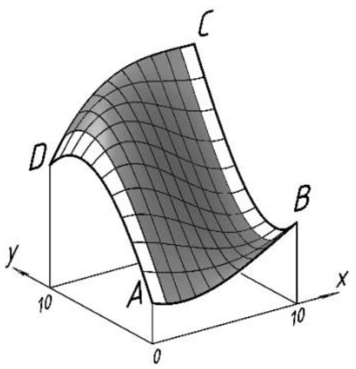


Fig. 2. Bicubic portion

4. Bicubic band

A bicubic band consists of series-connected (with C^2 smoothness) bicubic portions. The bicubic portions *ABCD* and *BMNC* must be connected along the joint *BC* (Fig. 3).

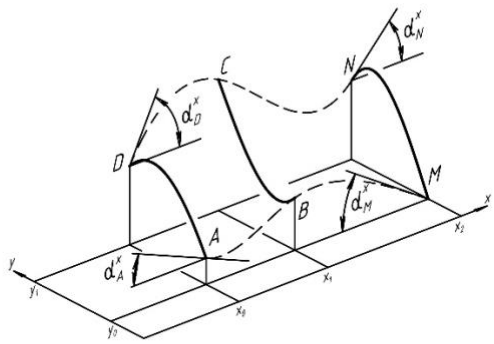


Fig. 3. Connection of bicubic portions (Theorem 1)

Let us show that C^2 smoothness is achieved if the longitudinal boundaries of the band are C^2-smooth and the first and second mixed derivatives at the junction points *B, C* of the connected portions are equal.

Theorem 1. To achieve C^2-smooth connection of the bicubic portions $\Phi_1(x, y)=ABCD$ and $\Phi_2(x, y)=BMNC$ along the transverse joint *BC*, it is sufficient to ensure that the longitudinal boundaries *ABM* and *DCN* are C^2-smooth and that the first and second mixed derivatives are equal at points *B, C*:

$$\frac{\partial^2\Phi_1}{\partial x\partial y}\Big|_B=\frac{\partial^2\Phi_2}{\partial x\partial y}\Big|_B,\quad \frac{\partial^2\Phi_1}{\partial x\partial y}\Big|_C=\frac{\partial^2\Phi_2}{\partial x\partial y}\Big|_C, \tag{10}$$

$$\frac{\partial^4\Phi_1}{\partial x^2\partial y^2}\Big|_B=\frac{\partial^4\Phi_2}{\partial x^2\partial y^2}\Big|_B,\quad \frac{\partial^4\Phi_1}{\partial x^2\partial y^2}\Big|_C=\frac{\partial^4\Phi_2}{\partial x^2\partial y^2}\Big|_C. \tag{11}$$

Proof. The requirement for C^2-smoothness of the band Φ1+Φ2 means that at any point of the joint *BC* the following equalities should be met:

$$\frac{\partial\Phi_1}{\partial x}=\frac{\partial\Phi_2}{\partial x}, \tag{12}$$

$$\frac{\partial^2\Phi_1}{\partial x^2}=\frac{\partial^2\Phi_2}{\partial x^2}. \tag{13}$$

Let us consider condition (12). The cubic functions $\phi_1(y)=\frac{\partial\Phi_1}{\partial x},\ \phi_2(y)=\frac{\partial\Phi_2}{\partial x}$ included in (12) are uniquely determined by their values at the boundary points of the joint *B, C*, as well as by the values of the first derivatives $\frac{\partial\phi_1}{\partial y}=\frac{\partial^2\Phi_1}{\partial x\partial y},\ \frac{\partial\phi_2}{\partial y}=\frac{\partial^2\Phi_2}{\partial x\partial y}$ at these points. According to (10), these values coincide; therefore, the functions $\phi_1(y),\ \phi_2(y)$ coincide along the joint *BC*. Condition (12) is satisfied.

Let us consider condition (13). The cubic functions $\psi_1(y)=\frac{\partial^2\Phi_1}{\partial x^2},\ \psi_2(y)=\frac{\partial^2\Phi_2}{\partial x^2}$ included in (13) are uniquely determined either by their values and first derivatives at points *B* and *C*, or by their values and second derivatives at these points. Therefore, to fulfill requirement (13), in addition to the equality of the functions $\psi_1(y),\ \psi_2(y)$ at points *B* and *C*, we should additionally ensure either the equality of the first derivatives $\frac{\partial \psi_1}{\partial y}=\frac{\partial^3 \Phi_1}{\partial x^2 \partial y}, \quad \frac{\partial \psi_2}{\partial y}=\frac{\partial^3 \Phi_2}{\partial x^2 \partial y}$, or the equality of the second derivatives $\frac{\partial^2 \psi_1}{\partial y^2}=\frac{\partial^4 \Phi_1}{\partial x^2 \partial y^2}, \quad \frac{\partial^2 \psi_2}{\partial y^2}=\frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}$ at these points. According to (11), the second derivatives at points B, C coincide. According to the C^2-smoothness condition for boundary curves ABM and DCN, the values of the functions $\psi_1(y), \psi_2(y)$ at points B and C also coincide. Therefore, the functions $\psi_1(y), \psi_2(y)$ coincide along the joint BC. Condition (13) is satisfied. The theorem is proven.

Let us construct a two-section C^2-smooth bicubic band with the fixed transverse guides AD, BC, MN and with the given gradients $\alpha_A^x, \alpha_M^x, \alpha_N^x, \alpha_D^x$ in the longitudinal direction (see Fig. 3). According to the condition of Theorem 1, the longitudinal boundary ABM formed by the cubic parabolas AB and BM should be a C^2-smooth compound curve (cubic spline). The same requirement applies to the longitudinal boundary DCN. Let us consider an algorithm for constructing a cubic spline with fixed end points (fixed tangents at the finite points).

4.1. Cubic spline with fixed end points

Points $A(x_0, y_0, z_0)$, $B(x_1, y_0, z_1)$, $M(x_2, y_0, z_2)$ are indicated in the vertical plane $y=y_0$ of the Cartesian coordinate system xyz. A composite C^2-smooth curve formed by the cubic parabolas $f_1(x)=AB$ and $f_2(x)=BM$ (cubic spline) must be drawn through these points. The slope angles α_A^x, α_M^x of the tangents to the constructed curve ("fixed end points") are indicated at boundary points A and M.

The condition for the C^2-smooth connection of the parabolas $f_1(x)$ and $f_2(x)$ has the form [11, 12]

$$h_{1x}S_0 + 2(h_{1x}+h_{2x})S_1 + h_{2x}S_2 = \frac{6(z_2-z_1)}{h_{2x}} - \frac{6(z_1-z_0)}{h_{1x}}, \tag{14}$$

where S_0, S_1, and S_2 are the values of the second derivatives of the functions $z=f_1(x)$ and $z=f_2(x)$ at the nodes A, B, *and* M. The designations $h_{1x}=x_1-x_0$ and $h_{2x}=x_2-x_1$ are used hereinafter.

Condition (14) should be supplemented with the fixed end points conditions:

$$\begin{aligned} &\frac{z_1-z_0}{h_{1x}} - \frac{h_{1x}}{3}S_0 - \frac{h_{1x}}{6}S_1 = tg\alpha_A^x, \\ &\frac{z_2-z_1}{h_{2x}} + \frac{h_{2x}}{3}S_2 + \frac{h_{2x}}{6}S_1 = tg\alpha_M^x. \end{aligned} \tag{15}$$

We find the values of S_0, S_1, S_2 from the system of equations (14), (15) and substitute them into the equations

$$\begin{aligned} f_1(x) &= \frac{S_0(x_1-x)^3 + S_1(x-x_0)^3}{6h_{1x}} + \left(\frac{z_0}{h_{1x}} - \frac{S_0 h_{1x}}{6}\right)(x_1-x) + \left(\frac{z_1}{h_{1x}} - \frac{S_1 h_{1x}}{6}\right)(x-x_0), \quad x\in[x_0, x_1], \\ f_2(x) &= \frac{S_1(x_2-x)^3 + S_2(x-x_1)^3}{6h_{2x}} + \left(\frac{z_1}{h_{2x}} - \frac{S_1 h_{2x}}{6}\right)(x_2-x) + \left(\frac{z_2}{h_{2x}} - \frac{S_2 h_{2x}}{6}\right)(x-x_1), \quad x\in[x_1, x_2]. \end{aligned} \tag{16}$$

The computational algorithm of (14), (15), (16) makes it possible to find equations of a cubic spline with fixed end points. If the spline is formed from m segments (m-1 junction points), the specified algorithm will contain m-1 smoothness conditions of form (14) and m equations of form (16).

4.2. Algorithm for calculating a bicubic tape

Let us construct a band consisting of *m* bicubic portions *ABCD*, *BMNC*, *MKLN*, ... (Fig. 4).

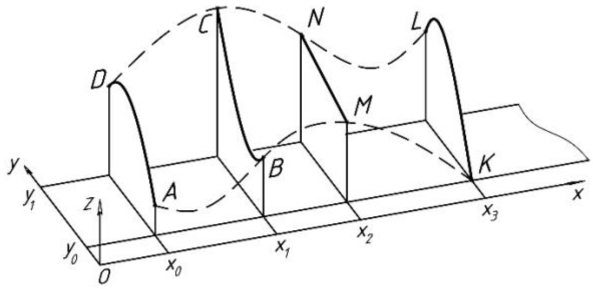


Fig. 4. Transverse guides of the bicubic band

We will assume that the equations of the frame transverse lines *AD, BC, MN, KL*, ... are either preset or found according to (8). To solve the problem, 16*m* coefficients included in equations (1) of connected portions must be calculated.

Step 1. We find the equations of the longitudinal boundaries of the band formed by composite C^2-smooth *m*-sectional cubic curves (cubic splines) using algorithm (14) ... (16).

Step 2. We create a system of 5*m* equations with (6). We supplement this system of equations with four boundary conditions with (7) plane angles and 4(*m*-1) smoothness conditions of (10) and (11) (see Theorem 1). We obtain a system of 9*m* linear equations, from which we find 9*m* coefficients.

Step 3. Using direct calculation by formulas (4) and (5), we find 7*m* coefficients included in the equations of portions. Jointly with the previously found 9*m* coefficients, we obtain 16*m* coefficients included in the equations of connected portions. The problem is solved.

Example 2

Let us construct a C^2-smooth bicubic band passing through fixed transverse guides

$$\begin{aligned} &AD = z_{AD}(y) = 2.5 + 1.7y - 0.09y^2 - 0.003y^3, \quad x = 0, \quad y \in [0,10], \\ &BC = z_{BC}(y) = 5 - y + 0.255y^2 - 0.008y^3, \quad x = 10, \quad y \in [0,10], \\ &MN = z_{MN}(y) = 2y - 0.015y^2 - 0.011y^3, \quad x = 25, \quad y \in [0,10]. \end{aligned} \tag{17}$$

Longitudinal gradients $tg\alpha_A^x = -0,4;$ $tg\alpha_D^x = 1,7;$ $tg\alpha_M^x = -0,6;$ $tg\alpha_N^x = 1$ are set in the angular points (see Fig. 3).

Solution

We will find the equation of the portion Φ_1=*ABCD* in form (1a). We will find the equation of the portion Φ_2=*BMNC* in the form

$$\begin{aligned} &\Phi_2(x, y) = b_{00} + b_{01}(y - y_0) + b_{02}(y - y_0)^2 + b_{03}(y - y_0)^3 + \\ &+[b_{10} + b_{11}(y - y_0) + b_{12}(y - y_0)^2 + b_{13}(y - y_0)^3](x - x_1) + \\ &+[b_{20} + b_{21}(y - y_0) + b_{22}(y - y_0)^2 + b_{23}(y - y_0)^3](x - x_1)^2 + \\ &+[b_{30} + b_{31}(y - y_0) + b_{32}(y - y_0)^2 + b_{33}(y - y_0)^3](x - x_1)^3, \; x \in [x_1, \; x_2], \, y \in [y_0, \; y_1], \end{aligned} \tag{1b}$$

where $x_0=y_0=0$, $x_1=y_1=10$, $x_2=25$.

Step 1. According to (14), (15), and (16), we find the equations for the segments of the longitudinal boundary *ABM* connected at point *B* with C^2 smoothness:

$$AB = z_{AB}(x) = 2.5 - 0.4x + 0.1285x^2 - 0.00635x^3, \quad x \in [0,10], \quad y = y_0 = 0,$$

$$BM = z_{BM}(x) = 5 + 0.265(x - 10) - 0.062(x - 10)^2 + 0.001474074(x - 10)^3, \quad x \in [10, 25], \quad y = y_0 = 0.$$

Similarly, we find the equations of the segments of boundary curve *DCN* connected at point *C* with C^2 smoothness:

$$DC = z_{DC}(x) = 7.5 + 1.7x - 0.144x^2 + 0.0024x^3, \quad x \in [0,10], \quad y = y_1 = 10,$$

$$CN = z_{CN}(x) = 12.5 - 0.46(x-10) - 0.072(x-10)^2 + 0.00536296(x-10)^3, \quad x \in [10, 25], \quad y = y_1 = 10.$$

Step 2. We make a system of ten equations of form (6) with respect to the unknown coefficients a_{ij}, b_{ij} included in equations (1a), (1b) of the required bicubic portions:

$$\begin{aligned}
&a_{11}h_{1x} + a_{21}h_{1x}^2 + a_{31}h_{1x}^3 = \beta_{BC} - \beta_{AD}, && b_{11}h_{2x} + b_{21}h_{2x}^2 + b_{31}h_{2x}^3 = \beta_{MN} - \beta_{BC},\\
&a_{12}h_{1x} + a_{22}h_{1x}^2 + a_{32}h_{1x}^3 = \gamma_{BC} - \gamma_{AD}, && b_{12}h_{2x} + b_{22}h_{2x}^2 + b_{32}h_{2x}^3 = \gamma_{MN} - \gamma_{BC},\\
&a_{13}h_{1x} + a_{23}h_{1x}^2 + a_{33}h_{1x}^3 = \delta_{BC} - \delta_{AD}, && b_{13}h_{2x} + b_{23}h_{2x}^2 + b_{33}h_{2x}^3 = \delta_{MN} - \delta_{BC},\\
&a_{11}h_y + a_{12}h_y^2 + a_{13}h_y^3 = \beta_{DC} - \beta_{AB}, && b_{11}h_y + b_{12}h_y^2 + b_{13}h_y^3 = \beta_{CN} - \beta_{BM},\\
&a_{21}h_y + a_{22}h_y^2 + a_{23}h_y^3 = \gamma_{DC} - \gamma_{AB}, && b_{21}h_y + b_{22}h_y^2 + b_{23}h_y^3 = \gamma_{CN} - \gamma_{BM}.
\end{aligned} \tag{18}$$

Here, $h_y = y_1 - y_0 = 10$, $h_{1x} = x_1 - x_0 = 10$, $h_{2x} = x_2 - x_1 = 15$.

We write down the plane angles condition:

$$\frac{\partial^2 \Phi_1}{\partial x \partial y}|_A = a_{11} = 0, \quad \frac{\partial^2 \Phi_1}{\partial x \partial y}|_D = 2a_{12} + 3h_y a_{13} = 0, \quad \frac{\partial^2 \Phi_2}{\partial x \partial y}|_M = b_{11} + 2h_{2x}b_{21} + 3h_{2x}^2 b_{31} = 0,$$
$$\frac{\partial^2 \Phi_2}{\partial x \partial y}|_N = 2b_{12} + 3h_y b_{13} + 4h_{2x}b_{22} + 6h_{2x}h_y b_{23} + 6h_{2x}^2 b_{32} + 9h_{2x}^2 h_y b_{33} = 0. \tag{19}$$

The system of equations (18) and (19) is supplemented by the requirements for a C^2-smooth connection of the bicubic portions Φ_1, Φ_2 along the joint *BC* (see Theorem 1):

$$\frac{\partial^2 \Phi_1}{\partial x \partial y}|_B = \frac{\partial^2 \Phi_2}{\partial x \partial y}|_B \Rightarrow a_{11} + 2h_{1x}a_{21} + 3h_{1x}^2 a_{31} = b_{11},$$
$$\frac{\partial^2 \Phi_1}{\partial x \partial y}|_C = \frac{\partial^2 \Phi_2}{\partial x \partial y}|_C \Rightarrow 2a_{12} + 3h_y a_{13} + 4h_{1x}a_{22} + 6h_{1x}h_y a_{23} + 6h_{1x}^2 a_{32} + 9h_{1x}^2 h_y a_{33} = 2b_{12} + 3h_y b_{13}, \tag{20}$$
$$\frac{\partial^4 \Phi_1}{\partial x^2 \partial y^2}|_B = \frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}|_B \Rightarrow a_{22} + 3h_{1x}a_{32} = b_{22}, \quad \frac{\partial^4 \Phi_1}{\partial x^2 \partial y^2}|_C = \frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}|_C \Rightarrow a_{23} + 3h_{1x}a_{33} = b_{33}.$$

We obtained a system of eighteen equations (18), (19), and (20). We find 18 coefficients from this system of equations:

$$a_{11} = 0;\ a_{12} = 0.063;\ a_{13} = -0.0042;\ a_{21} = -0.0687;\ a_{22} = -0.001425;$$
$$a_{23} = 0.000557;\ a_{31} = 0.00417;\ a_{32} = -0.0001425;\ a_{33} = -0.0000187,$$
$$b_{11} = -0.123;\ b_{12} = -0.00825;\ b_{13} = 0.00133;\ b_{21} = 0.0564;\ b_{22} = -0.0057;$$
$$b_{23} = -0.000004;\ b_{31} = -0.002324(4);\ b_{32} = 0.000336(6);\ b_{33} = -0.00000653(3).$$

Step 3. We find the remaining 14 coefficients according to (4) and (5):

$$a_{00} = \alpha_{AB} = 2{,}5; \quad a_{10} = \beta_{AB} = -0{,}4; \quad a_{20} = \gamma_{AB} = 0{,}1285; \quad a_{30} = \delta_{AB} = -0{,}00635;$$
$$a_{01} = \beta_{AD} = 1{,}7; \quad a_{02} = \gamma_{AD} = -0{,}09; \quad a_{03} = \delta_{AD} = -0{,}003;$$
$$b_{00} = \alpha_{BM} = 5; \quad b_{10} = \beta_{BM} = 0{,}265; \quad b_{20} = \gamma_{BM} = -0{,}062; \quad b_{30} = \delta_{BM} = 0{,}001474074;$$
$$b_{01} = \beta_{BC} = -1; \quad b_{02} = \gamma_{BC} = 0{,}255; \quad b_{03} = \delta_{BC} = -0{,}008.$$

We determined all the coefficients of equations (1a), (1b) of the bicubic segments Φ_1 and Φ_2 with a C^2-smooth connection. Figure 5 shows the grid of generators of the bicubic band $\Phi_1 + \Phi_2$ constructed according to (1a) and (1b).

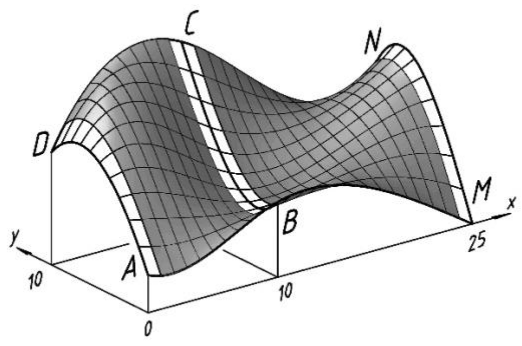
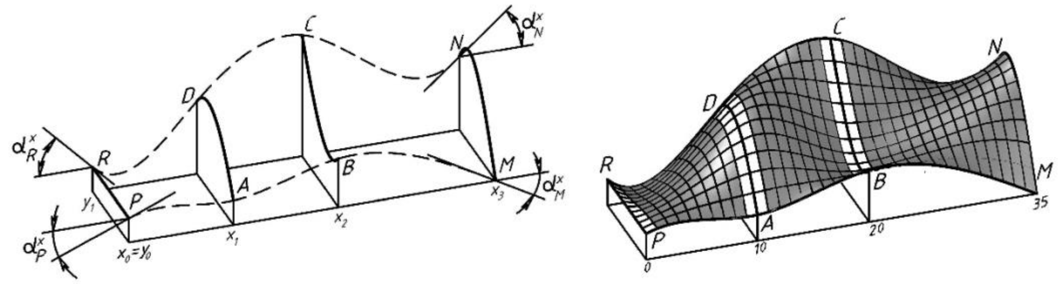


Fig. 5. C2-smooth band (Example 2)

Example 3

Let us attach the section *PADR* with a horizontal guide *PR* located at a height of z=2.5 to the two-section band *ABCD* + *BMN* considered in example 2 (Fig. 6a).



(a) (b)

Fig. 6. Three-section C2-smooth band (Example 3):
a - boundary conditions; b - grid of generators

We will assume we are given the coordinates of the grid nodal points: x_0=y_0=0, x_1=y_1=10, x_2=20, x_3=35. The equations for the transverse guides *AD, BC, MN* are shown in example 2. The equation for the guide line *PR* degenerates into the equation z_{PR} =2.5. The longitudinal gradients $tg\alpha_P^x = 0.4;\ tg\alpha_M^x = -0.6;\ tg\alpha_N^x = 1;\ tg\alpha_R^x = -1$ are set at the angular points *P, M, N, R* of the constructed band.

Solution

Let us find the equations of the bicubic portions Φ_1=*PADR* and Φ_2=*ABCD* with (1a) and (1b), assuming that x_0=y_0=0, x_1= y_1=10, x_2=20 and the equation of the portion Φ_3=*BMNC* in the following form

$$\begin{aligned}\Phi_3(x,y) &= c_{00} + c_{01}(y-y_0) + c_{02}(y-y_0)^2 + c_{03}(y-y_0)^3 + \\ &+[c_{10} + c_{11}(y-y_0) + c_{12}(y-y_0)^2 + c_{13}(y-y_0)^3](x-x_2) + \\ &+[c_{20} + c_{21}(y-y_0) + c_{22}(y-y_0)^2 + c_{23}(y-y_0)^3](x-x_2)^2 + \\ &+[c_{30} + c_{31}(y-y_0) + c_{32}(y-y_0)^2 + c_{33}(y-y_0)^3](x-x_2)^3,\ \ x\in[x_2,\ x_3],\ y\in[y_0,\ y_1],\end{aligned} \tag{1c}$$

assuming that x_3=35.

Step 1. According to algorithm (14) ... (16), we find the equations of the segments of the longitudinal boundary *PABM*. We obtain a set of C^2-smoothly connected cubic parabolas satisfying the "fixed end points" conditions:

$$\begin{aligned}&PA = z_{PA}(x) = 2.5 + 0.4x - 0.08554053x^2 + 0.004554053x^3, \quad x\in[0,10], \quad y = y_0 = 0, \\ &AB = z_{AB}(x) = 2.5 + 0.055405613(x-10) + 0.05108106(x-10)^2 - 0.003162161(x-10)^3, \quad x\in[10,20], \\ &BM = z_{BM}(x) = 5 + 0.128378513(x-20) - 0.04378370(x-20)^2 + 0.000866867(x-20)^3, \quad x\in[20,35].\end{aligned} \tag{21}$$

Using the same algorithm, we find the equations for the segments of the longitudinal boundary *RDCN*:

$$
\begin{aligned}
&RD = z_{RD}(x) = 2.5 - x + 0.2432432x^2 - 0.009319729x^3, \quad x \in [0,10], \quad y = y_1 = 10,\\
&DC = z_{DC}(x) = 7.5 + 1.0675675(x-10) - 0.03648648(x-10)^2 - 0.00202027027(x-10)^3, \quad x \in [10, 20],\\
&CN = z_{CN}(x) = 12.5 - 0.2702702(x-20) - 0.097297306(x-20)^2 + 0.006206206(x-20)^3, \quad x \in [20, 35].
\end{aligned} \tag{22}
$$

Step 2. We make a system of $5m$=15 equations of form (6) with respect to a_{ij}, b_{ij}, and c_{ij}:

$$
\begin{aligned}
&a_{11}h_{1x} + a_{21}h_{1x}^2 + a_{31}h_{1x}^3 = \beta_{AD} - \beta_{PR}, \quad b_{11}h_{2x} + b_{21}h_{2x}^2 + b_{31}h_{2x}^3 = \beta_{BC} - \beta_{AD}, \quad c_{11}h_{3x} + c_{21}h_{3x}^2 + c_{31}h_{3x}^3 = \beta_{MN} - \beta_{BC},\\
&a_{12}h_{1x} + a_{22}h_{1x}^2 + a_{32}h_{1x}^3 = \gamma_{AD} - \gamma_{PR}, \quad b_{12}h_{2x} + b_{22}h_{2x}^2 + b_{32}h_{2x}^3 = \gamma_{BC} - \gamma_{AD}, \quad c_{12}h_{3x} + c_{22}h_{3x}^2 + c_{32}h_{3x}^3 = \gamma_{MN} - \gamma_{BC},\\
&a_{13}h_{1x} + a_{23}h_{1x}^2 + a_{33}h_{1x}^3 = \delta_{AD} - \delta_{PR}, \quad b_{13}h_{2x} + b_{23}h_{2x}^2 + b_{33}h_{2x}^3 = \delta_{BC} - \delta_{AD}, \quad c_{13}h_{3x} + c_{23}h_{3x}^2 + c_{33}h_{3x}^3 = \delta_{MN} - \delta_{BC},\\
&a_{11}h_y + a_{12}h_y^2 + a_{13}h_y^3 = \beta_{RD} - \beta_{PA}, \quad b_{11}h_y + b_{12}h_y^2 + b_{13}h_y^3 = \beta_{DC} - \beta_{AB}, \quad c_{11}h_y + c_{12}h_y^2 + c_{13}h_y^3 = \beta_{CN} - \beta_{BM},\\
&a_{21}h_y + a_{22}h_y^2 + a_{23}h_y^3 = \gamma_{RD} - \gamma_{PA}, \quad b_{21}h_y + b_{22}h_y^2 + b_{23}h_y^3 = \gamma_{DC} - \gamma_{AB}, \quad c_{21}h_y + c_{22}h_y^2 + c_{23}h_y^3 = \gamma_{CN} - \gamma_{BM}.
\end{aligned} \tag{23}
$$

The values β, γ, δ included in (23) are determined according to (17), (21), and (22). For example, it follows from (17) that $\beta_{AD} = 1.7;\ \gamma_{AD} = -0.09;\quad \delta_{AD} = -0.003$. Line *PR* is a horizontal segment, so $\beta_{PR} = \gamma_{PR} = \delta_{PR} = 0$.

We supplement the system of equations (23) with the conditions of form (7) (plane angles):

$$
\begin{aligned}
&\frac{\partial^2 \Phi_1}{\partial x \partial y}\Big|_P = a_{11} = 0, \quad \frac{\partial^2 \Phi_1}{\partial x \partial y}\Big|_R = 2a_{12} + 3h_y a_{13} = 0, \quad \frac{\partial^2 \Phi_3}{\partial x \partial y}\Big|_M = c_{11} + 2h_{3x}c_{21} + 3h_{3x}^2 c_{31} = 0,\\
&\frac{\partial^2 \Phi_3}{\partial x \partial y}\Big|_N = 2c_{12} + 3h_y c_{13} + 4h_{3x}c_{22} + 6h_{3x}h_y c_{23} + 6h_{3x}^2 c_{32} + 9h_{3x}^2 h_y c_{33} = 0.
\end{aligned} \tag{24}
$$

We write down $4(m$-1)=8 smoothness conditions: the conditions for the equality of the first mixed derivatives

$$
\begin{aligned}
&\frac{\partial^2 \Phi_1}{\partial x \partial y}\Big|_A = \frac{\partial^2 \Phi_2}{\partial x \partial y}\Big|_A \Rightarrow a_{11} + 2h_{1x}a_{21} + 3h_{1x}^2 a_{31} = b_{11},\\
&\frac{\partial^2 \Phi_1}{\partial x \partial y}\Big|_D = \frac{\partial^2 \Phi_2}{\partial x \partial y}\Big|_D \Rightarrow 2a_{12} + 3h_y a_{13} + 4h_{1x}a_{22} + 6h_{1x}h_y a_{23} + 6h_{1x}^2 a_{32} + 9h_{1x}^2 h_y a_{33} = 2b_{12} + 3h_y b_{13},\\
&\frac{\partial^2 \Phi_2}{\partial x \partial y}\Big|_B = \frac{\partial^2 \Phi_3}{\partial x \partial y}\Big|_B \Rightarrow b_{11} + 2h_{2x}b_{21} + 3h_{2x}^2 b_{31} = c_{11},\\
&\frac{\partial^2 \Phi_2}{\partial x \partial y}\Big|_C = \frac{\partial^2 \Phi_3}{\partial x \partial y}\Big|_C \Rightarrow 2b_{12} + 3h_y b_{13} + 4h_{2x}b_{22} + 6h_{2x}h_y b_{23} + 6h_{2x}^2 b_{32} + 9h_{2x}^2 h_y b_{33} = 2c_{12} + 3h_y c_{13},
\end{aligned} \tag{25}
$$

and the conditions for the equality of the second mixed derivatives

$$
\begin{aligned}
&\frac{\partial^4 \Phi_1}{\partial x^2 \partial y^2}\Big|_A = \frac{\partial^2 \Phi_2}{\partial x \partial y}\Big|_A \Rightarrow\ a_{22} + 3h_{1x}a_{32} = b_{22}, \quad \frac{\partial^4 \Phi_1}{\partial x^2 \partial y^2}\Big|_D = \frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}\Big|_D \Rightarrow a_{23} + 3h_{1x}a_{33} = b_{33},\\
&\frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}\Big|_B = \frac{\partial^4 \Phi_3}{\partial x^2 \partial y^2}\Big|_B \Rightarrow\ b_{22} + 3h_{2x}b_{32} = c_{22}, \quad \frac{\partial^4 \Phi_2}{\partial x^2 \partial y^2}\Big|_C = \frac{\partial^4 \Phi_3}{\partial x^2 \partial y^2}\Big|_C \Rightarrow b_{23} + 3h_{2x}b_{33} = c_{33}
\end{aligned} \tag{26}
$$

at the junction points *A, D, B,* and *C* (see Theorem 1).

We find 27 coefficients from the system of 27 linear equations (23) ... (26):

$a_{11} = 0; a_{12} = -0.042; a_{13} = 0.0028; a_{21} = 0.05115056; a_{22} = 0.0027851339;$
$a_{23} = -0.00046123472; a_{31} = -0.0034150506; a_{32} = 0.000051486612; a_{33} = 0.000015123472;$
$b_{11} = -0.0015050574; b_{12} = 0.029148637; b_{13} = -0.0018876512; b_{21} = -0.073461158; b_{22} = 0.0043297322;$
$b_{23} = 0.00021407081; b_{31} = 0.0046611663; b_{32} = -0.00037945959; b_{33} = -0.000007530569;$
$c_{11} = -0.07237831; c_{12} = 0.0019053939; c_{13} = 0.000134537; c_{21} = 0.049650441; c_{22} = -0.0070540555;$
$c_{23} = 0.000155387535; c_{31} = -0.0020994592; c_{32} = 0.00038180195; c_{33} = -0.00001184626.$

Step 3. We find the remaining coefficients using the formulas of (4), (5):

$a_{00} = \alpha_{PA} = 2.5; a_{10} = \beta_{PA} = 0.4; a_{20} = \gamma_{PA} = -0.08554053; a_{30} = \delta_{PA} = 0.004554053;$
$a_{01} = \beta_{PR} = 0; a_{02} = \gamma_{PR} = 0; a_{03} = \delta_{PR} = 0;$
$b_{00} = \alpha_{AB} = 2.5; b_{10} = \beta_{AB} = 0.055405613; b_{20} = \gamma_{AB} = 0.05108106; b_{30} = \delta_{AB} = -0.003162161;$
$b_{01} = \beta_{AD} = 1.7; b_{02} = \gamma_{AD} = -0.09; b_{03} = \delta_{AD} = -0.003;$
$c_{00} = \alpha_{BM} = 5; c_{10} = \beta_{BM} = 0.128378513; c_{20} = \gamma_{BM} = -0.043783701; c_{30} = \delta_{BM} = 0.000866867;$
$c_{01} = \beta_{BC} = -1; c_{02} = \gamma_{BC} = 0.255; c_{03} = \delta_{BC} = -0.008.$

We determined all 48 coefficients of equations (1a), (1b), and (1c) of the bicubic portions Φ_1, Φ_2, and Φ_3. Figure 6b shows the grid of generators of the bicubic band $\Phi=\Phi_1+\Phi_2+\Phi_3$.

5. Bicubic surface

Let us construct a bicubic surface consisting of *mn* bicubic portions: *m* portions in the longitudinal direction (along the *x* axis) and *n* portions in the transverse direction (along the *y* axis). The longitudinal and transverse frame lines of the surface are formed by cubic splines. We will assume that the constructed surface consists of *n* bicubic bands, each of which consists of *m* bicubic portions. The bicubic bands should be C^2-smoothly interconnected (along the longitudinal lines of the frame). Let us consider the conditions for the smooth connection of bicubic bands.

Let the bicubic surface consist of two bands (Fig. 7).

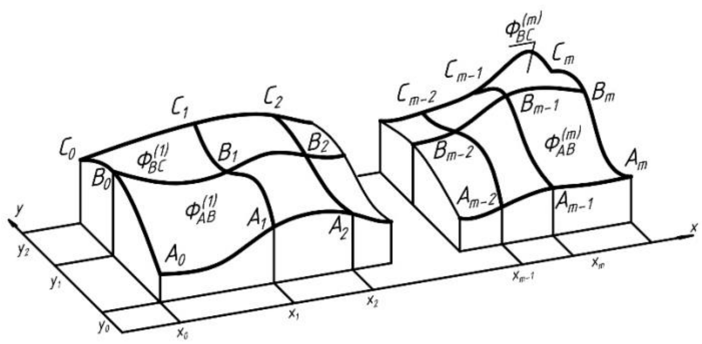


Fig. 7. Bicubic surface frame

Band Φ_{AB} formed by the bicubic portions $\Phi_{AB}^{(1)}, \Phi_{AB}^{(2)}, \ldots, \Phi_{AB}^{(m)}$ is bounded by the cubic splines $A_0A_1 \ldots A_m$ and $B_0B_1 \ldots B_m$. B and Φ_{BC} formed by the bicubic portions $\Phi_{BC}^{(1)}, \Phi_{BC}^{(2)}, \ldots, \Phi_{BC}^{(m)}$ is bounded by the cubic splines $B_0B_1 \ldots B_m$ and $C_0C_1 \ldots C_m$.

Theorem 2. To achieve C^2-smooth connection of the bicubic bands Φ_{AB}, Φ_{BC} along the longitudinal joint $B_0 \ldots B_m$, it is sufficient to ensure the equality of the first and second mixed derivatives at the initial B_0 and finite B_m points of the longitudinal joint in addition to the C^2-smoothness of the frame lines:

$$\frac{\partial^2\Phi_{AB}^{(1)}}{\partial x\partial y}|_{B_0}=\frac{\partial^2\Phi_{BC}^{(1)}}{\partial x\partial y}|_{B_0},\quad \frac{\partial^2\Phi_{AB}^{(m)}}{\partial x\partial y}|_{B_m}=\frac{\partial^2\Phi_{BC}^{(m)}}{\partial x\partial y}|_{B_m},\tag{27}$$

$$\frac{\partial^4\Phi_{AB}^{(1)}}{\partial x^2\partial y^2}|_{B_0}=\frac{\partial^4\Phi_{BC}^{(1)}}{\partial x^2\partial y^2}|_{B_0},\quad \frac{\partial^4\Phi_{AB}^{(m)}}{\partial x^2\partial y^2}|_{B_m}=\frac{\partial^4\Phi_{BC}^{(m)}}{\partial x^2\partial y^2}|_{B_m}.\tag{28}$$

Proof. The requirement for the C^2-smooth connection of bicubic bands means that the following equalities are met at any point of the joint $B_0 \ldots B_m$

$$\frac{\partial\Phi_{AB}}{\partial y}=\frac{\partial\Phi_{BC}}{\partial y},\tag{29}$$

$$\frac{\partial^2\Phi_{AB}}{\partial y^2}=\frac{\partial^2\Phi_{BC}}{\partial y^2}.\tag{30}$$

Let us consider condition (29). The cubic C^2-smooth composite functions $\phi_{AB}(x)=\frac{\partial\Phi_{AB}}{\partial y}$, $\phi_{BC}(x)=\frac{\partial\Phi_{BC}}{\partial y}$ included (29) are determined by their values at nodal points B_0, B_1, ..., B_m and the values of the first derivatives $\frac{\partial^2\Phi}{\partial x\partial y}$ at finite points B_0, B_m of the longitudinal joint $B_0 \ldots B_m$. The numerical values of the functions $\phi_{AB}(x)$, $\phi_{BC}(x)$ at points B_0, B_1, ..., B_m are equal to the tangents of the slope angles of the tangents to the transverse lines of the frame. Due to the smoothness of the transverse lines of the frame, the values of the functions $\phi_{AB}(x)$, $\phi_{BC}(x)$ at these points coincide. According to condition (27), the values of the derivatives of these functions at the finite points B_0 and B_m also coincide; therefore, the functions $\phi_{AB}(x)$, $\phi_{BC}(x)$ coincide along the joint $B_0 \ldots. B_m$. Condition (29) is satisfied.

Let us consider condition (30). The cubic C^2-smooth composite functions $\psi_{AB}(x)=\frac{\partial^2\Phi_{AB}}{\partial y^2}$, $\psi_{BC}(x)=\frac{\partial^2\Phi_{BC}}{\partial y^2}$ included in (30) are determined by their values at points B_0, B_1, ..., B_m and the values of the second derivatives $\frac{\partial^2}{\partial x^2}(\frac{\partial^2\Phi}{\partial y^2})$ at boundary points B_0, B_m of the longitudinal joint $B_0 \ldots B_m$. The values of the functions $\psi_{AB}(x)$, $\psi_{BC}(x)$ at points B_0, B_1, ..., B_m are proportional to the curvature of the transverse joints at these points. Due to the C^2-smoothness of the transverse lines of the frame, these values coincide. According to condition (28), the values of the second derivatives of these functions at boundary points B_0, B_m also coincide; therefore, the functions $\psi_{AB}(x)$, $\psi_{BC}(x)$ coincide along the joint $B_0 \ldots. B_m$. Condition (30) is satisfied. The theorem is proven.

5.1. Algorithm for calculating a bicubic surface

The calculation is reduced to calculating $16mn$ coefficients of the equations of the bicubic portions forming the constructed surface. We assume that the surface consists of n longitudinal bicubic bands, and each band consists of m bicubic portions. Each portion is described by an equation of form (1).

Step 1. Using the algorithm from (14), (15), and (16), we find the equations of the longitudinal and transverse lines of the frame (cubic splines).

Step 2. We make a system of $5m$ equations of form (6) for each bicubic band. We add $4(m-1)$ smoothness conditions for the band (see Theorem 1). We obtain $9m-4$ equations. We obtain $n(9m-4)$ equations for n bands. We supplement the resulting equations with $4(n-1)$ conditions (27), (28) for the smooth band connection (see Theorem 2), as well as 4 "plane angles" conditions. We obtain the system of $9mn$ linear equations used to find $9mn$ coefficients included in the equations of bicubic portions of the constructed surface.

Step 3. Using direct calculation of formulas (4) and (5), we find $7mn$ coefficients included in the equations of portions. Along with the previously found $9mn$ coefficients, we obtain $16mn$ coefficients included in the equations of connected portions. The problem is solved.

Example 4

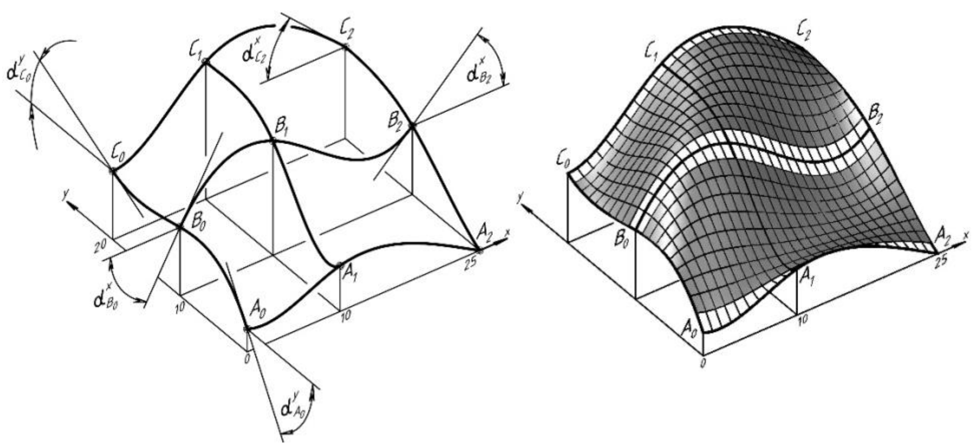
Let us construct a C^2-smooth bicubic surface passing through points

$$A_0(0;0;2,5),\quad A_1(10;0;5),\quad A_2(25;0;0),$$
$$B_0(0;10;7,5),\quad B_1(10;10;12,5),\quad B_2(25;10;7,5),$$
$$C_0(0;20;7,5),\quad C_1(10;20;15),\quad C_2(25;20;10).$$

The gradients in the longitudinal and transverse directions are fixed at the angular points A_0, A_2, C_0, C_2:

$$tg\alpha_{A0}^x = -0.4;\quad tg\alpha_{A0}^y = 1.7;\quad tg\alpha_{A2}^x = -0.6;\quad tg\alpha_{A2}^y = 0.5;$$
$$tg\alpha_{C0}^x = 0;\quad tg\alpha_{C0}^y = 0.5;\quad tg\alpha_{C2}^x = 1;\quad tg\alpha_{C2}^y = 0.$$

The longitudinal gradients $tg\alpha_{B0}^x = 1,7;\ tg\alpha_{B2}^x = 1$ are given at boundary points B_0, B_2. The transverse gradients $tg\alpha_{A1}^y = -1;\ tg\alpha_{C1}^y = 0$ are given at boundary points A_1, C_1 (Fig. 8a).



(a) (b)

Fig. 8. Bicubic surface (Example 4): a - fixed frame; b - grid of generators

Solution

Step 1. Per (14), (15), and (16) we find the equations of the frame lines satisfying the conditions of the problem.

The equation of the longitudinal boundary line $A_0A_1A_2$:

$$\begin{aligned} &A_0A_1 = 2.5 - 0.4x + 0.1285x^2 - 0.00635x^3,\quad x \in [0,10],\quad y = y_0 = 0,\\ &A_1A_2 = 5 + 0.265(x-10) - 0.062(x-10)^2 + 0.001474074(x-10)^3,\quad x \in [10, 25],\quad y = y_0 = 0. \end{aligned} \tag{31}$$

The equation of the longitudinal line $B_0B_1B_2$:

$$\begin{aligned} &B_0B_1 = 7.5 + 1.7x - 0.144x^2 + 0.0024x^3,\quad x \in [0,10],\quad y = y_1 = 10,\\ &B_1B_2 = 12.5 - 0.46(x-10) - 0.072(x-10)^2 + 0.00536296(x-10)^3,\quad x \in [10, 25],\quad y = y_1 = 10. \end{aligned} \tag{32}$$

The equation of the longitudinal boundary line $C_0C_1C_2$:

$$\begin{aligned} &C_0C_1 = 7.5 + 0.1575x^2 - 0.00825x^3,\quad x \in [0,10],\quad y = y_2 = 20,\\ &C_1C_2 = 15 + 0.675(x-10) - 0.09(x-10)^2 + 0.001518518519(x-10)^3,\quad x \in [10, 25],\quad y = y_2 = 20. \end{aligned} \tag{33}$$

(31) ... (33) take into account that $x_0=y_0=0$, $x_1=10$, $x_2=25$.

Similarly, we find the equations of the transverse lines of the frame. The equation of the transverse boundary line $A_0B_0C_0$:

$$A_0B_0 = 2.5+1.7y-0.1725y^2+0.00525y^3, \quad y\in[0,10], \quad x=x_0=0,$$
$$B_0C_0 = 7.5-0.175(y-10)-0.015(y-10)^2+0.00325(y-10)^3, \quad y\in[10,20], \quad x=x_0=0. \tag{34}$$

The equation of the transverse line $A_1B_1C_1$:

$$A_1B_1 = 5-y+0,325y^2-0,015y^3, \quad y\in[0,10], \quad x=x_1=10,$$
$$B_1C_1 = z_{CN}(x) = 12,5+(y-10)-0,125(y-10)^2+0,005(y-10)^3, \quad y\in[10,20], \quad x=x_1=10. \tag{35}$$

The equation of the transverse boundary line $A_2B_2C_2$:

$$A_2B_2 = 0.5y+0.0625y^2-0.0375y^3, \quad y\in[0,10], \quad x=x_2=25,$$
$$B_2C_2 = 7.5+0.625(y-10)-0.05(y-10)^2+0.00125(y-10)^3, \quad y\in[10,20], \quad x=x_2=25. \tag{36}$$

The surface frame is fixed.

Step 2. The constructed surface consists of two bicubic bands Φ_{AB} and Φ_{BC} smoothly interconnected along the joint $B_0B_1B_2$. The band Φ_{AB} is formed by the portions

$$\Phi_{AB}^{(1)} = A_0A_1B_1B_0 = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}(x-x_0)^3(y-y_0)^3, \quad \Phi_{AB}^{(2)} = A_1A_2B_2B_1 = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{ij}(x-x_1)^3(y-y_0)^3. \tag{37}$$

The band Φ_{BC} is formed by the portions

$$\Phi_{BC}^{(1)} = B_0B_1C_1C_0 = \sum_{i=0}^{3}\sum_{j=0}^{3} c_{ij}(x-x_0)^3(y-y_1)^3, \quad \Phi_{BC}^{(2)} = B_1B_2C_2C_1 = \sum_{i=0}^{3}\sum_{j=0}^{3} d_{ij}(x-x_1)^3(y-y_1)^3. \tag{38}$$

We make a system of 10 equations of form (6) for each band.

For the band Φ_{AB} we obtain:

$$\begin{aligned}
&a_{11}h_{1x}+a_{21}h_{1x}^2+a_{31}h_{1x}^3 = \beta_{A_1B_1}-\beta_{A_0B_0}, && b_{11}h_{2x}+b_{21}h_{2x}^2+b_{31}h_{2x}^3 = \beta_{A_2B_2}-\beta_{A_1B_1},\\
&a_{12}h_{1x}+a_{22}h_{1x}^2+a_{32}h_{1x}^3 = \gamma_{A_1B_1}-\gamma_{A_0B_0}, && b_{12}h_{2x}+b_{22}h_{2x}^2+b_{32}h_{2x}^3 = \gamma_{A_2B_2}-\gamma_{A_1B_1},\\
&a_{13}h_{1x}+a_{23}h_{1x}^2+a_{33}h_{1x}^3 = \delta_{A_1B_1}-\delta_{A_0B_0}, && b_{13}h_{2x}+b_{23}h_{2x}^2+b_{33}h_{2x}^3 = \delta_{A_2B_2}-\delta_{A_1B_1},\\
&a_{11}h_{1y}+a_{12}h_{1y}^2+a_{13}h_{1y}^3 = \beta_{B_0B_1}-\beta_{A_0A_1}, && b_{11}h_{1y}+b_{12}h_{1y}^2+b_{13}h_{1y}^3 = \beta_{B_1B_2}-\beta_{A_1A_2},\\
&a_{21}h_{1y}+a_{22}h_{1y}^2+a_{23}h_{1y}^3 = \gamma_{B_0B_1}-\gamma_{A_0A_1}, && b_{21}h_{1y}+b_{22}h_{1y}^2+b_{23}h_{1y}^3 = \gamma_{B_1B_2}-\gamma_{A_1A_2}.
\end{aligned} \tag{39}$$

Here, $h_{1x}=x_1\text{-}x_0=10$, $h_{1y}=y_1\text{-}y_0=10$, and $h_{2x}=x_2\text{-}x_1=15$.

For the band Φ_{BC} we obtain:

$$\begin{aligned}
&c_{11}h_{1x}+c_{21}h_{1x}^2+c_{31}h_{1x}^3 = \beta_{B_1C_1}-\beta_{B_0C_0}, && d_{11}h_{2x}+d_{21}h_{2x}^2+d_{31}h_{2x}^3 = \beta_{B_2C_2}-\beta_{B_1C_1},\\
&c_{12}h_{1x}+c_{22}h_{1x}^2+c_{32}h_{1x}^3 = \gamma_{B_1C_1}-\gamma_{B_0C_0}, && d_{12}h_{2x}+d_{22}h_{2x}^2+d_{32}h_{2x}^3 = \gamma_{B_2C_2}-\gamma_{B_1C_1},\\
&c_{13}h_{1x}+c_{23}h_{1x}^2+c_{33}h_{1x}^3 = \delta_{B_1C_1}-\delta_{B_0C_0}, && d_{13}h_{2x}+d_{23}h_{2x}^2+d_{33}h_{2x}^3 = \delta_{B_2C_2}-\delta_{B_1C_1},\\
&c_{11}h_{2y}+c_{12}h_{2y}^2+c_{13}h_{2y}^3 = \beta_{C_0C_1}-\beta_{B_0B_1}, && d_{11}h_{2y}+d_{12}h_{2y}^2+d_{13}h_{2y}^3 = \beta_{C_1C_2}-\beta_{B_1B_2},\\
&c_{21}h_{2y}+c_{22}h_{2y}^2+c_{23}h_{2y}^3 = \gamma_{C_0C_1}-\gamma_{B_0B_1}, && d_{21}h_{2y}+d_{22}h_{2y}^2+d_{23}h_{2y}^3 = \gamma_{C_1C_2}-\gamma_{B_1B_2}.
\end{aligned} \tag{40}$$

Here, h_{2y}=10. The values β, γ, δ included in (39) and (40) are determined according to (31)...(36). For example, it follows from (34) that $\beta_{A_0B_0}=1.7; \gamma_{A_0B_0}=-0.1725; \delta_{A_0B_0}=0.00525.$

We write down the smoothness conditions for each bicubic band (see Theorem 1). For band Φ_{AB}, we obtain four conditions for a smooth connection of portions $\Phi_{AB}^{(1)}, \Phi_{AB}^{(2)}$ (the equality of the first and second mixed derivatives at points A_1, B_1):

$$\frac{\partial^2\Phi_{AB}^{(1)}}{\partial x\partial y}|_{A_1}=\frac{\partial^2\Phi_{AB}^{(2)}}{\partial x\partial y}|_{A_1}\Rightarrow a_{11}+2h_{1x}a_{21}+3h_{1x}^2a_{31}=b_{11},$$
$$\frac{\partial^2\Phi_{AB}^{(1)}}{\partial x\partial y}|_{B_1}=\frac{\partial^2\Phi_{AB}^{(2)}}{\partial x\partial y}|_{B_1}\Rightarrow 2a_{12}+3h_{1y}a_{13}+4h_{1x}a_{22}+6h_{1x}h_{1y}a_{23}+6h_{1x}^2a_{32}+9h_{1x}^2h_{1y}a_{33}=2b_{12}+3h_{1y}b_{13}, \quad (41)$$
$$\frac{\partial^4\Phi_{AB}^{(1)}}{\partial x^2\partial y^2}|_{A_1}=\frac{\partial^4\Phi_{AB}^{(2)}}{\partial x^2\partial y^2}|_{A_1}\Rightarrow a_{22}+3h_{1x}a_{32}=b_{22},\quad \frac{\partial^4\Phi_{AB}^{(1)}}{\partial x^2\partial y^2}|_{B_1}=\frac{\partial^4\Phi_{AB}^{(2)}}{\partial x^2\partial y^2}|_{B_1}\Rightarrow a_{23}+3h_{1x}a_{33}=b_{33}.$$

For band Φ_{BC} we also obtain four conditions for a smooth connection of portions $\Phi_{BC}^{(1)},\Phi_{BC}^{(2)}$ (the equality of the first and second mixed derivatives at points B_1, C_1):

$$\frac{\partial^2\Phi_{BC}^{(1)}}{\partial x\partial y}|_{B_1}=\frac{\partial^2\Phi_{BC}^{(2)}}{\partial x\partial y}|_{B_1}\Rightarrow c_{11}+2h_{1x}c_{21}+3h_{1x}^2a_{31}=d_{11},$$
$$\frac{\partial^2\Phi_{BC}^{(1)}}{\partial x\partial y}|_{C_1}=\frac{\partial^2\Phi_{BC}^{(2)}}{\partial x\partial y}|_{C_1}\Rightarrow 2c_{12}+3h_{2y}c_{13}+4h_{1x}a_{22}+6h_{1x}h_{2y}a_{23}+6h_{1x}^2a_{32}+9h_{1x}^2h_{2y}a_{33}=2d_{12}+3h_{2y}d_{13}, \quad (42)$$
$$\frac{\partial^4\Phi_{BC}^{(1)}}{\partial x^2\partial y^2}|_{B_1}=\frac{\partial^4\Phi_{BC}^{(2)}}{\partial x^2\partial y^2}|_{B_1}\Rightarrow c_{22}+3h_{1x}c_{32}=d_{22},\quad \frac{\partial^4\Phi_{BC}^{(1)}}{\partial x^2\partial y^2}|_{C_1}=\frac{\partial^4\Phi_{BC}^{(2)}}{\partial x^2\partial y^2}|_{C_1}\Rightarrow c_{23}+3h_{1x}c_{33}=d_{33}.$$

We write down the conditions for the smooth connection of the bicubic bands Φ_{AB} and Φ_{BC} at the junction points B_0 and B_2 (see Theorem 2):

$$\frac{\partial^2\Phi_{AB}^{(1)}}{\partial x\partial y}|_{B_0}=\frac{\partial^2\Phi_{BC}^{(1)}}{\partial x\partial y}|_{B_0}\Rightarrow a_{11}+2h_{1y}a_{12}+3h_{1y}^2a_{13}=c_{11},\quad \frac{\partial^4\Phi_{AB}^{(1)}}{\partial x^2\partial y^2}|_{B_0}=\frac{\partial^4\Phi_{BC}^{(1)}}{\partial x^2\partial y^2}|_{B_0}\Rightarrow a_{22}+3h_{1y}a_{23}=c_{22},$$
$$\frac{\partial^2\Phi_{AB}^{(2)}}{\partial x\partial y}|_{B_2}=\frac{\partial^2\Phi_{BC}^{(2)}}{\partial x\partial y}|_{B_2}\Rightarrow b_{11}+2h_{1y}b_{12}+3h_{1y}^2b_{13}+2h_{2x}(b_{21}+2h_{1y}b_{22}+3h_{1y}^2b_{23})+3h_{2x}^2(b_{31}+2h_{1y}b_{32}+3h_{1y}^2b_{33})= \quad (43)$$
$$=d_{11}+2h_{2x}d_{21}+3h_{2x}^2d_{31},\quad \frac{\partial^4\Phi_{AB}^{(2)}}{\partial x^2\partial y^2}|_{B_2}=\frac{\partial^4\Phi_{BC}^{(2)}}{\partial x^2\partial y^2}|_{B_2}\Rightarrow b_{22}+3h_{1y}b_{23}+3h_{2x}b_{32}+9h_{2x}h_{1y}b_{33}=d_{22}+3h_{2x}d_{32}.$$

We write down the plane angles condition:

$$\frac{\partial^2\Phi_{AB}^{(1)}}{\partial x\partial y}|_{A_0}=a_{11}=0,\quad \frac{\partial^2\Phi_{AB}^{(2)}}{\partial x\partial y}|_{A_2}=b_{11}+h_{2x}b_{21}+3h_{2x}^2b_{31}=0,\quad \frac{\partial^2\Phi_{BC}^{(1)}}{\partial x\partial y}|_{C_0}=c_{11}+2h_{2y}c_{12}+3h_{2y}^2c_{13}=0,$$
$$\frac{\partial^2\Phi_{BC}^{(2)}}{\partial x\partial y}|_{C_2}=d_{11}+2h_{2y}d_{12}+3h_{2y}^2d_{13}+2h_{2x}(d_{21}+2h_{2y}d_{22}+3h_{2y}^2d_{23})+3h_{2x}^2(d_{31}+2h_{2y}d_{32}+3h_{2y}^2d_{33})=0. \quad (44)$$

The system of equations (39) ... (44) contains 36 equations with respect to 36 coefficients a_{ij}, b_{ij}, c_{ij}, and d_{ij} included in equations (37) and (38). Solving this system of equations, we obtain:

$a_{11}=0;\ a_{12}=0.06;\ a_{13}=-0.0039;\ a_{21}=-0.0627;\ a_{22}=0.0023175;\ a_{23}=0.00012275;\ a_{31}=0.00357;\ a_{32}=-0.00033425;$
$a_{33}=0.00000647;\ b_{11}=-0.183;\ b_{12}=0.006075;\ b_{13}=0.0004975;\ b_{21}=0.0444;\ b_{22}=-0.00771;\ b_{23}=0.000317;$
$b_{31}=-0.001702(2);\ b_{32}=0.000409(2);\ b_{33}=-0.00002001(1);\ c_{11}=0.03;\ c_{12}=-0.057;\ c_{13}=0.0037;\ c_{21}=0.020475;$
$c_{22}=0.006;\ c_{23}=-0.0005032;\ c_{31}=-0.0011725;\ c_{32}=-0.00014;\ c_{33}=0.000015075;\ d_{11}=0.08775;\ d_{12}=0.021;$
$d_{13}=-0.001842;\ d_{21}=-0.0147;\ d_{22}=0.0018;\ d_{23}=-0.000051;\ d_{31}=0.000478(8);\ d_{32}=-0.000191(1);\ d_{33}=0.00001047(7).$

Step 3. Using direct calculation of formulas (4) and (5), we find the remaining 28 coefficients included in equations (37), (38):

$a_{00}=\alpha_{A_0A_1}=2.5;\ a_{10}=\beta_{A_0A_1}=-0.4;\ a_{20}=\gamma_{A_0A_1}=0.1285;\ a_{30}=\delta_{A_0A_1}=-0.00635;\ a_{01}=\beta_{A_0B_0}=1.7;\ a_{02}=\gamma_{A_0B_0}=-0.1725;$
$a_{03}=\delta_{A_0B_0}=0.00525;\ b_{00}=\alpha_{A_1A_2}=5;\ b_{10}=\beta_{A_1A_2}=0.265;\ b_{20}=\gamma_{A_1A_2}=-0.062;\ b_{30}=\delta_{A_1A_2}=0.001474074;\ b_{01}=\beta_{A_1B_1}=-1;$
$b_{02}=\gamma_{A_1B_1}=0.325;\ b_{03}=\delta_{A_1B_1}=-0.015;\ c_{00}=\alpha_{B_0B_1}=7.5;\ c_{10}=\beta_{B_0B_1}=1.7;\ c_{20}=\gamma_{B_0B_1}=-0.144;\ c_{30}=\delta_{B_0B_1}=0.0024;$
$c_{01}=\beta_{B_0C_0}=-0.175;\ c_{02}=\gamma_{B_0C_0}=-0.015;\ c_{03}=\delta_{B_0C_0}=0.00325;\ d_{00}=\alpha_{B_1B_2}=12.5;\ d_{10}=\beta_{B_1B_2}=-0.46;\ d_{20}=\gamma_{B_1B_2}=-0.072;$
$d_{30}=\delta_{B_1B_2}=0.00536296(296);\ d_{01}=\beta_{B_1C_1}=1;\ d_{02}=\gamma_{B_1C_1}=-0.125;\ d_{03}=\delta_{B_1C_1}=0.005.$

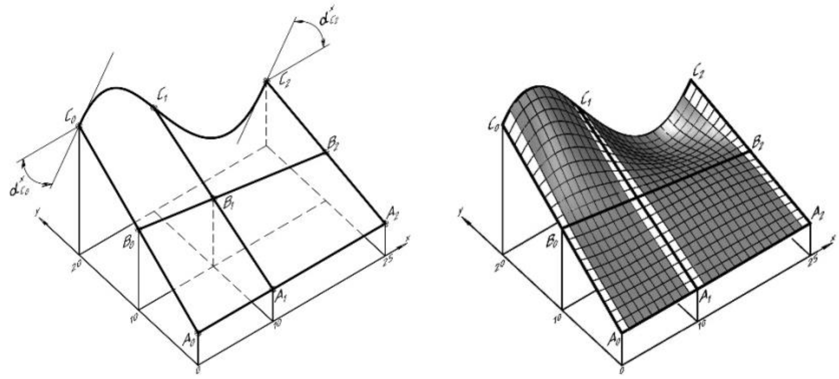
We determined all 64 coefficients included in equations (37) and (38). Figure 8b shows the grid of the generators of the bicubic surface Φ_{AB}+Φ_{BC}.

5.2. Private cases of a bicubic surface

The computational algorithm based on preliminary fixing of the surface frame is valid when the frame is formed by a mixed set of cubic splines.

Example 5

The surface frame is set by the straight lines $B_0B_1B_2$ and $A_1B_1C_1$. The surface boundaries are set by the straight lines $A_0A_1A_2$, $A_0B_0C_0$, and $A_2B_2C_2$ and the cubic spline $C_0C_1C_2$ with the longitudinal gradients $tg\alpha_{C_0}^{x} = tg\alpha_{C_2}^{x} = 2$ (Fig. 9a).



(a) (b)

Fig. 9. Bicubic surface on a frame of straight lines and cubic splines (Example 5):
a - fixed frame; b - grid of generators

We are given the coordinates of the following nodal points:

$A_0(0;\,0;\,5),\quad A_1(10;\,0;\,5),\quad A_2(25;\,0;\,5),$

$B_0(0;\,10;\,7{,}5),\quad B_1(10;\,10;\,12{,}5),\quad B_2(25;\,10;\,7{,}5),$

$C_0(0;\,20;\,7{,}5),\quad C_1(10;\,20;\,15),\quad C_2(25;\,20;\,10).$

Let us construct a C^2-smooth bicubic surface satisfying the conditions of the problem.

Solution

We will assume that the constructed surface consists of the bicubic bands Φ_{AB} and Φ_{BC} smoothly interconnected along the straight joint $B_0B_1B_2$. Each band, in turn, consists of bicubic portions (37), (38).

Step 1. We find the equations of the frame lines satisfying the conditions of the problem.

The equation of line $A_0A_1A_2$:

$$A_0A_1 = A_1A_2 = z(x) = 5, \quad x \in [0, 25], \quad y = y_0 = 0.$$

The equation of line $B_0B_1B_2$:

$$B_0B_1 = 12.5 - 0.2x, \quad x \in [0, 10], \quad y = y_1 = 10,$$

$$B_1B_2 = 10.5 - 0.2(x - 10), \quad x \in [10, 25], \quad y = y_1 = 10.$$

The equation of line $A_0B_0C_0$:

$$A_0B_0 = 5 + 0.75y, \quad y \in [0, 10], \quad x = x_0 = 0,$$

$$B_0C_0 = 12.5 + 0.75(y - 10), \quad y \in [10, 20], \quad x = x_0 = 0.$$

The equation of line $A_1B_1C_1$:

$$A_1B_1 = 5 + 0.55y, \quad y \in [0, 10], \quad x = x_1 = 10,$$

$$B_1C_1 = 10.5 + 0.55(y - 10), \quad y \in [10, 20], \quad x = x_1 = 10.$$

The equation of line $A_2B_2C_2$:

$$A_2B_2 = 5 + 0.25y, \quad y \in [0,10], \quad x = x_2 = 25,$$
$$B_2C_2 = 7.5 + 0.25(y-10), \quad y \in [10, 20], \quad x = x_2 = 25.$$

Following the algorithm in (14), (15), and (16), we find the equation of the C²-smooth boundary curve $C_0C_1C_2$ with the gradients $tg\alpha^x_{C_0} = tg\alpha^x_{C_2} = 2$:

$$C_0C_1 = 20 + 2x - 0.36x^2 + 0.012x^3, \quad x \in [0,10], \quad y = y_2 = 20,$$
$$C_1C_2 = 16 - 1.6(x-10) + 0.09(x-10)^2 + 0.0053(3)(x-10)^3, \quad x \in [10, 25], \quad y = y_2 = 20.$$

Step 2. We write down the system of equations (39) ... (44) containing 36 equations for 36 coefficients a_{ij}, b_{ij}, c_{ij}, and d_{ij} included in equations (37) and (38). In equations (39) and (40) we substitute the values of the coefficients α, β, γ, and δ corresponding to the equations of frame lines. For example, it follows from the equation of boundary line $A_0B_0C_0$ that $\alpha_{A_0B_0} = 5; \beta_{A_0B_0} = 0.75; \gamma_{A_0B_0} = \delta_{A_0B_0} = 0.$

Having solved the system of equations (39) ... (44), we find 36 coefficients included in equations (37), (38):

$$a_{11} = 0; a_{12} = -0.021; a_{13} = 0.0019; a_{21} = -0.003; a_{22} = 0.00315; a_{23} = -0.000285; a_{31} = 0.0001; a_{32} = -0.000105;$$
$$a_{33} = 0.0000095; b_{11} = -0.03; b_{12} = 0.0105; b_{13} = -0.00095; b_{21} = 0; b_{22} = 0; b_{23} = 0; b_{31} = 0.00004(4);$$
$$b_{32} = -0.000046(6); b_{33} = 0.0000042(2); \ c_{11} = 0.15; c_{12} = 0.036; c_{13} = -0.0029; c_{21} = -0.0255;$$
$$c_{22} = -0.0054; c_{23} = 0.000435; c_{31} = 0.00085; c_{32} = 0.00018; c_{33} = -0.0000145; d_{11} = -0.105; d_{12} = -0.018;$$
$$d_{13} = 0.00145; d_{21} = 0; d_{22} = 0; d_{23} = 0; d_{31} = 0.00037(7); d_{32} = 0.00008; d_{33} = -0.0000064(4).$$

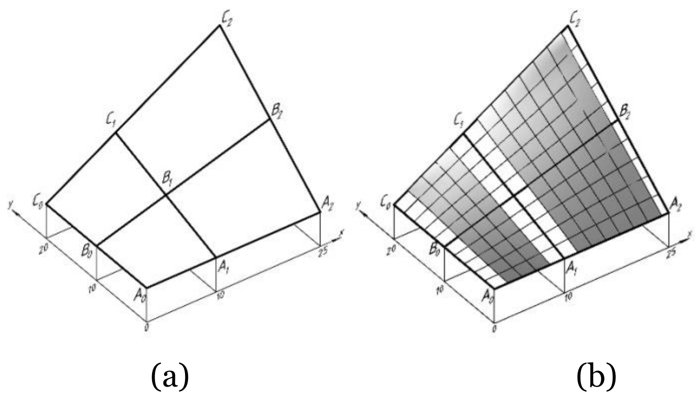
Step 3. Using direct calculation of formulas (4) and (5), we find the remaining 28 coefficients included in equations (37) and (38):

$$a_{00} = \alpha_{A_0A_1} = 5; a_{10} = \beta_{A_0A_1} = 0; a_{20} = \gamma_{A_0A_1} = 0; a_{30} = \delta_{A_0A_1} = 0; a_{01} = \beta_{A_0B_0} = 0.75; a_{02} = \gamma_{A_0B_0} = 0;$$
$$a_{03} = \delta_{A_0B_0} = 0; b_{00} = \alpha_{A_1A_2} = 5; b_{10} = \beta_{A_1A_2} = 0; b_{20} = \gamma_{A_1A_2} = 0; b_{30} = \delta_{A_1A_2} = 0; b_{01} = \beta_{A_1B_1} = 0.55;$$
$$b_{02} = \gamma_{A_1B_1} = 0; b_{03} = \delta_{A_1B_1} = 0; c_{00} = \alpha_{B_0B_1} = 12.5; c_{10} = \beta_{B_0B_1} = -0.2; c_{20} = \gamma_{B_0B_1} = 0; c_{30} = \delta_{B_0B_1} = 0;$$
$$c_{01} = \beta_{B_0C_0} = 0.75; c_{02} = \gamma_{B_0C_0} = 0; c_{03} = \delta_{B_0C_0} = 0; d_{00} = \alpha_{B_1B_2} = 10.5; d_{10} = \beta_{B_1B_2} = -0.2; d_{20} = \gamma_{B_1B_2} = 0;$$
$$d_{30} = \delta_{B_1B_2} = 0; d_{01} = \beta_{B_1C_1} = 0.55; d_{02} = \gamma_{B_1C_1} = 0; d_{03} = \delta_{B_1C_1} = 0.$$

We determined all 64 coefficients included in equations (37) and (38). Figure 9b shows the grid of generators of the bicubic surface $\Phi_{AB}+\Phi_{BC}$.

Example 6

The surface frame is set by a spatial quadrilateral with the angular points $A_0(0; 0; 5)$, $A_2(25; 0; 5)$, $C_2(25; 20; 20)$, and $C_0(0; 20; 5)$ and the straight guides $B_0B_1B_2$ and $A_1B_1C_1$ lying in the vertical planes y=10 and x=10 (Fig. 10a).



(a) (b)

Fig. 10. Bicubic surface on a straight line frame (Example 6):
a - fixed frame; b - grid of generators

Let us construct a bicubic surface “stretched” on a given frame.

Solution

We will assume that the constructed surface consists of the bicubic bands Φ_{AB} and Φ_{BC} smoothly interconnected along the straight joint $B_0B_1B_2$. Each band, in turn, consists of bicubic portions (37), (38).

Step 1. We find the equations of the frame lines.

The equation of line $A_0A_1A_2$:

$$A_0A_1 = A_1A_2 = z(x) = 5, \quad x \in [0, 25], \quad y = y_0 = 0.$$

The equation of line $B_0B_1B_2$:

$$B_0B_1 = 5 + 0.3x, \quad x \in [0, 10], \quad y = y_1 = 10,$$

$$B_1B_2 = 8 + 0.3(x - 10), \quad x \in [10, 25], \quad y = y_1 = 10.$$

The equation of line $C_0C_1C_2$:

$$C_0C_1 = 5 + 0.6x, \quad x \in [0, 10], \quad y = y_1 = 10,$$

$$C_1C_2 = 11 + 0.6(x - 10), \quad x \in [10, 25], \quad y = y_1 = 10.$$

The equation of line $A_0B_0C_0$:

$$A_0B_0 = B_0C_0 = z(y) = 5, \quad y \in [0, 20], \quad x = x_0 = 0.$$

The equation of line $A_1B_1C_1$:

$$A_1B_1 = 5 + 0.3y, \quad y \in [0, 10], \quad x = x_1 = 10,$$

$$B_1C_1 = 8 + 0.3(y - 10), \quad y \in [10, 20], \quad x = x_1 = 10.$$

The equation of line $A_2B_2C_2$:

$$A_2B_2 = 5 + 0.75y, \quad y \in [0, 10], \quad x = x_2 = 25,$$

$$B_2C_2 = 12.5 + 0.75(y - 10), \quad y \in [10, 20], \quad x = x_2 = 25.$$

Step 2. We write down the system of equations (39) … (44) substituting values of the coefficients α, β, γ, δ corresponding to the equations of frame lines. For example, it follows from the equation of boundary line $A_0B_0C_0$ that $\alpha_{A_0B_0} = 5; \beta_{A_0B_0} = 0; \gamma_{A_0B_0} = \delta_{A_0B_0} = 0.$ Having solved the system of equations (39) … (44), we find the coefficients of equations (37) and (38):

$$a_{11} = 0; a_{12} = 0.0045; a_{13} = -0.00015; a_{21} = 0.0045; a_{22} = -0.000675; a_{23} = 0.0000225; a_{31} = -0.00015; a_{32} = 0.0000225;$$
$$a_{33} = -0.00000075; b_{11} = 0.045; b_{12} = -0.00225; b_{13} = 0.000075; b_{21} = 0; b_{22} = 0; b_{23} = 0; b_{31} = -0.00006(6);$$
$$b_{32} = 0.00001; b_{33} = -0.0000003(3); \; c_{11} = 0.045; c_{12} = 0; c_{13} = -0.00015; c_{21} = -0.00225;$$
$$c_{22} = 0; c_{23} = 0.0000225; c_{31} = 0.000075; c_{32} = 0; c_{33} = -0.00000075; d_{11} = 0.0225; d_{12} = 0;$$
$$d_{13} = 0.000075; d_{21} = 0; d_{22} = 0; d_{23} = 0; d_{31} = 0.00003(3); d_{32} = 0; d_{33} = -0.0000003(3).$$

Step 3. Using direct calculation of the formulas (4) and (5), we find the remaining 28 coefficients included in equations (37) and (38):

$$a_{00} = \alpha_{A_0A_1} = 5; a_{10} = \beta_{A_0A_1} = 0; a_{20} = \gamma_{A_0A_1} = 0; a_{30} = \delta_{A_0A_1} = 0; a_{01} = \beta_{A_0B_0} = 0; a_{02} = \gamma_{A_0B_0} = 0;$$
$$a_{03} = \delta_{A_0B_0} = 0; b_{00} = \alpha_{A_1A_2} = 5; b_{10} = \beta_{A_1A_2} = 0; b_{20} = \gamma_{A_1A_2} = 0; b_{30} = \delta_{A_1A_2} = 0; b_{01} = \beta_{A_1B_1} = 0.3;$$
$$b_{02} = \gamma_{A_1B_1} = 0; b_{03} = \delta_{A_1B_1} = 0; c_{00} = \alpha_{B_0B_1} = 5; c_{10} = \beta_{B_0B_1} = 0,3; c_{20} = \gamma_{B_0B_1} = 0; c_{30} = \delta_{B_0B_1} = 0;$$
$$c_{01} = \beta_{B_0C_0} = 0; c_{02} = \gamma_{B_0C_0} = 0; c_{03} = \delta_{B_0C_0} = 0; d_{00} = \alpha_{B_1B_2} = 8; d_{10} = \beta_{B_1B_2} = 0,3; d_{20} = \gamma_{B_1B_2} = 0;$$
$$d_{30} = \delta_{B_1B_2} = 0; d_{01} = \beta_{B_1C_1} = 0,3; d_{02} = \gamma_{B_1C_1} = 0; d_{03} = \delta_{B_1C_1} = 0.$$

We determined all the coefficients included in the equations of bicubic portions (37) and (38). Figure 10b shows the grid of the generators of the bicubic surface Φ_{AB}+Φ_{BC}. The resulting surface slightly differs from the oblique plane $A_0A_2C_2C_0$. For example, at x=6, y=4, the elevation marks of the points on the bicubic surface and on the oblique plane are z=5.623232 and z=5.720, respectively, differing by 1.7%.

6. Software

Matrix calculations (solving the systems of linear equations) were performed using the freely distributed software SMath Studio. The grid of the bicubic surface generators was calculated and visualized in all examples using the AutoLISP programming language in the AutoCAD environment [13]. The transparency of the examples is ensured by indicating the numerical values of all the calculated magnitudes with an accuracy of nine significant figures.

7. Discussion

We aimed to avoid the classical idea of a composite bicubic surface as a set of bicubic portions meeting certain conditions for the border of a surface (incidence of given points, fixed gradients, etc.) and for the smooth interconnection of portions.

Our proposed approach creates a surface frame made of algebraic cubic splines. Constructing a cubic spline that interpolates a given set of points is a trivial task which was fully solved in the second half of 20th century [14, 15]. The work of Kazan mathematicians Kornishin M.S. et. al [16] is noteworthy in Russian literature.

A distinctive feature of the proposed algorithm for calculating a composite C^2-smooth bicubic surface consists in the conventional decomposition of the constructed surface into separate bicubic tapes bounded by longitudinal frame lines. Calculating a tape is much easier than calculating a surface [17]. We believe that this approach, which divides the problem into simple calculation blocks, is most consistent with engineering practice.

The proposed algorithm was illustrated with 3D surface models. As P. Bezier said, the widespread use of some systems is adversely affected by the fact that, despite the sophistication of applied mathematical methods, users have difficulty in their assimilation [8]. One method to overcome these difficulties is to demonstrate the applied methods and algorithms based on specific examples, as we did in this article.

8. Conclusion

The paper proposes an algorithm for calculating a composite bicubic surface with a continuous change in curvature stretched on a fixed frame. In the proposed algorithm, the problem is divided into two stages: first, the frame equations are found, and then the coefficients included in the equations of the bicubic portions forming the bicubic surface are calculated. According to the specified boundary conditions, the frame lines are described by cubic splines with fixed end points.

This approach to modeling a bicubic surface reduces the size of the characteristic matrix of the system of linear equations with respect to the coefficients included in the bicubic surface equation. The matrix size is reduced from $16mn$ to $9mn$, where m and n are the number of bicubic portions along the x, y axes. Surface visualization is reduced to building a grid of longitudinal and transverse generators, the equations of which are formed from the bicubic surface equation by substituting y=const (longitudinal generators) or x=const (transverse generators).

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