

# Comparative Graphical Analysis of Distortions of Some Cartographic Projections

S.V. Ktitrov<sup>1</sup>, D.A. Ryslyayev<sup>2</sup>

National Research Nuclear University MEPhI  
(Moscow Engineering Physics Institute), Russian Federation

<sup>1</sup> ORCID: 0000-0002-7963-9151, [svktitrov@mephi.ru](mailto:svktitrov@mephi.ru)

<sup>2</sup> ORCID: 0000-0001-6781-1519, [rislaev97@gmail.com](mailto:rislaev97@gmail.com)

## **Abstract**

The article is devoted to the problem of distortion of the size and shape of extended objects on maps when projecting the Earth's surface. Various types of projections are considered, their classification is briefly given. The Mercator, Albers, Kavraisky and azimuthal equidistant projections were selected for analysis.

Distortions differ depending on the projection. It is pointed out that traditional graphical methods of distortion demonstration, such as distortion ellipses, cannot give a complete picture for extended objects without additional calculations. The authors propose an approach based on combining projected objects to solve the problem of rapid and visual detection of distortions.

Scale coefficients of projection methods were been analyzed to identify areas with the minimal distortion. Both expressions for coefficients and graphical dependences on latitude and longitude are given. It is proposed to use the Mercator projection when the figure is shifted to the equator region as a way to obtain the minimal distorted figure. Thus, a figure on a plane can be obtained. It is closest in shape and size to an object on the earth's surface.

Comparisons are carried out for various model objects (parallelogram, rectangle, circle, rhombus) located both near the equator and at the 60th parallel. The combination method demonstrates the distortion produced by various projection methods.

Visualization is performed by means of the author's software including the SINUS-D program for system simulation, as well as specially developed programs in C++ where preliminary data preparation, projection, displacement of objects are carried out and a program in Python that uses third-party libraries to display objects against the background of the globe.

The approach to the analysis of the representation of cartographic information proposed by the authors and implemented in the programs developed by the authors can be useful primarily in studying the features of cartographic projections, but it also has practical potential in everyday use to facilitate the planning of activities and more accurate accounting and allocation of resources.

**Keywords:** cartographic projection, visualization, comparative analysis, error, distortion, distortion ellipse, scale factor.

## **1. Introduction**

When solving the tasks of planning economic activities, navigation tasks, ensuring transport accessibility, eliminating the consequences of natural disasters, it is important to display the boundaries of regions, lines and distances on maps as accurately as possible. In the case when the areas under consideration are sufficiently extended, distortions and errors are inevitable due to the mapping of the spherical surface of the Earth onto the map plane. To date, a number of cartographic projections have been proposed that solve the problem of

minimizing errors according to various criteria. The advantages and disadvantages of projections are well known and described in detail in the literature [1,2]. However, general formulas and local differential characteristics are usually used to describe distortions. It does not allow the applied user of maps to easily estimate and predict the general integral errors that will arise in a particular case. The more detailed information presented in the article makes it easier to take into account distortion errors.

One of the features used in Kavraisky's classification is the nature of distortions [3]. According to the nature of distortion, projections are divided into:

1. Equiangular (conformal) - angles and azimuths are transmitted without distortion. As a result, similarity is preserved in such projections for infinitesimal parts. The cartographic grid in these projections is orthogonal. On maps you can measure angles and azimuths, and it is also convenient to measure distances on them in any direction.

2. Equal area (equivalent) - here the scale of areas always remains constant and equal to one, which means that areas are transmitted without distortion. On maps in such projections, you can make a comparison of areas.

3. Equidistant (equidistant) – here the scale in one of the main directions is preserved and is equal to one.

4. Arbitrary - there are all kinds of distortions. The essence of using such projections is the most uniform distribution of distortions on the map and the convenience of solving some practical problems.

This article discusses four projections that differ in the kind of distortion:

- Mercator Projection;
- Albers projection;
- Azimuthal equidistant projection;
- Kavraisky projection.

Taking into account the possibility of simultaneous presence of various types of distortions on the map, along with analytical dependences of distortions on geographic coordinates, it is convenient to use their graphic representation for the analysis of distortions, as well as displaying model figures in the projection methods under study. The last task is performed by means of the software developed by the authors.

## **2. The problem of traditional approaches to visualization of errors of cartographic projections**

In order to visually display the nature of distortion, distortion ellipses are used. The distortion ellipse (Tissot's indicatrix) is an infinitesimal ellipse, which is an image of an infinitesimal circle on the Earth's surface, with the help of which a generalized characteristic of the distortions of cartographic projections is performed.

This is a good way to visualize local differential characteristics, but it is possible to accurately convey these characteristics only with infinitely small ellipse sizes. In general, even for nearby points, the parameters of distortion ellipses may differ significantly. Nevertheless, the generally accepted approach is that ellipses are plotted on a conditional grid with a constant step, and the size of the ellipses is significant. Figure 1 shows the distortion ellipses for the Mercator projection. By the size and shape of the ellipses, it is visually possible to determine the relative scale factor and distortion in various directions at the point that is the center of the ellipse.

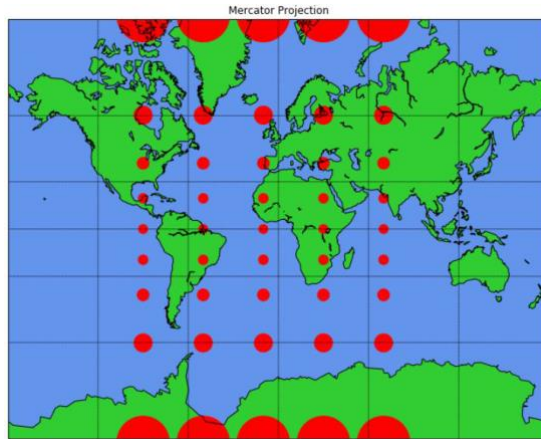


Fig 1. The Mercator projection with distortion ellipses

The convention of the size of the distortion ellipse can be misleading. Often, the size and shape of the distortion ellipses constructed for points on the boundary of the considered ellipse, or even for its interior points, can differ significantly from the considered one. We call an object extended, provided that the parameters of the distortion ellipses differ significantly for different points of the projected object for the problem in which the projection is used. The determining factors may not necessarily be the size of the object, but its location and method of projection.

Let's give an example. For the Mercator projection, the distortion ellipse map shows that the circles do not change shape, only their radius increases. Moreover, one can come across the idea of transforming a circle into a circle, which is true only in relation to infinitesimal radius. If such an interpretation is transferred to the case of circles of significant radius, errors are inevitable. So, a circle centered at the pole should be displayed in a straight line, and for a circle centered at the 45th parallel and with a radius of several thousand kilometers on the map, the part of the semicircle closer to the pole will have a noticeably larger area than the equatorial part. The deformation of a circle of considerable size is illustrated in Fig. 2. The yellow dot in Fig. 2 is the center of the circle. Accordingly, the difference in the areas of the upper and lower parts of the circle in the Mercator projection is clearly visible.

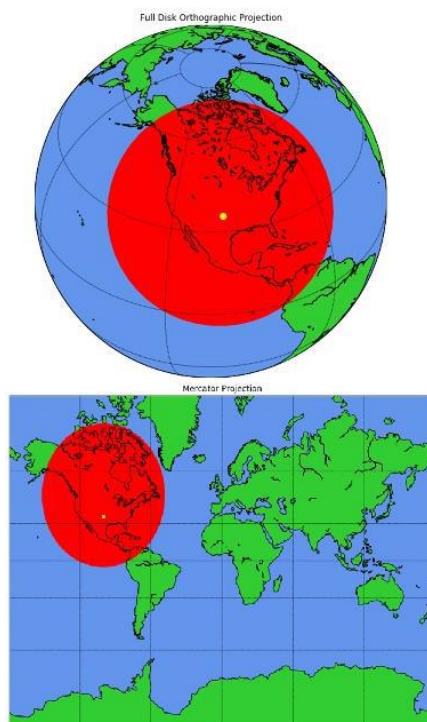


Fig.2 Large circles in the Mercator projection

Thus, the use of distortion ellipses cannot be the only way to initially visually assess distortion on a map. The problems of using distortion ellipses discussed above are well known to professional cartographers, however, the presence and quantitative nature of such distortions for most may come as a surprise or require complex calculations that are inappropriate under time constraints.

To overcome these difficulties, the authors propose to use a combination of the analysis of the dependence of the scale factors on coordinates with an approach to visualization based on the display of objects and areas that have finite dimensions that are essential to show distortions.

### 3. Analysis of scale factors

A scale factor is the ratio of an infinitesimal segment on the map to an infinitesimal segment on the projected surface [4]. The accuracy of representing the Earth as a ball is sufficient for the comparative analysis carried out, while the formulas become much simpler. In this case, the scale factors are calculated by the formulas

$$k = \frac{1}{R \cos \phi} \sqrt{\left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2}; \quad (1)$$

$$h = \frac{1}{R} \sqrt{\left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2}, \quad (2)$$

where  $R$  is the radius of the Earth,  $\phi$  – latitude,  $\lambda$  – longitude,  $X, Y$  – coordinates on the projection,  $k, h$  – scale factors along the parallel and meridian, respectively.

Let's analyze the distortions of some common projections using scale factors.

The Mercator projection is a conformal cylindrical projection that preserves the angles between directions. Coordinate transformation is performed according to the formulas:

$$X = R\lambda; \quad Y = R \ln \left( \operatorname{tg} \left( \frac{\phi}{2} + \frac{\pi}{4} \right) \right),$$

from where according to (1), (2) scale factors

$$k = h = \frac{1}{\cos \phi}.$$

The scale factors equality ensures that the projection is conformal. The scale increases towards the poles, reaching infinity on them. The radius of the distortion ellipses, which are circles, also changes proportionally to it (Fig. 1). The largest distortions in the size of objects appear near the poles (Fig. 3).

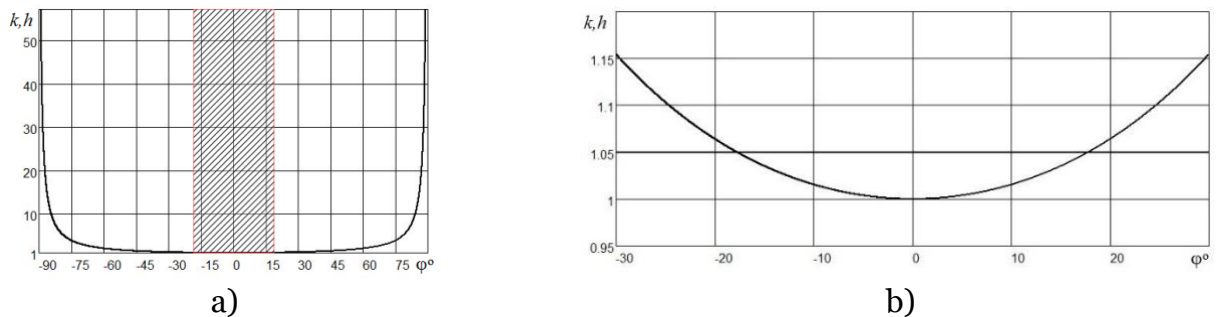


Fig. 3. Dependence of the scale factor on latitude for the Mercator projection

The scale factor increases approximately 57 times near the poles at the 89th parallel relative to the scale at the equator, where it is 1. The scale deviates from 1 by no more than 5% between the 18° parallels (Fig. 3a, marked by hatching, Fig. 3b shows larger).

Another approach to projection is used in the Albers projection. The Albers projection is a conic projection. Projection is carried out on the surface of a cone that cuts the Earth along two parallels. The top of the cone is located on the continuation of the earth's axis. The

parallels of a normal grid are represented by the arcs of concentric circles, and the meridians are their radii, the angles between which are proportional to the corresponding longitude differences. The Albers projection is used to display regions stretched in the latitudinal direction (from west to east). This projection preserves the area of the objects, but distorts the angles and shape of the contours. Projection is carried out according to the following formulas:

$$\begin{aligned} X &= p \sin \theta; \\ Y &= p_0 - p \cos \theta; \\ p &= \frac{1}{n} \sqrt{C - 2n \sin \phi}; \\ p_0 &= \frac{1}{n} \sqrt{C - 2n \sin \phi_0}; \\ \theta &= n(\lambda - \lambda_0); \\ C &= \cos^2 \phi_1 + 2n \sin \phi_1; \\ n &= \frac{1}{n(\sin \phi_1 + \sin \phi_2)}. \end{aligned}$$

Here  $\phi_0, \lambda_0$  is the latitude and longitude of the point that serves as the origin of coordinates in the projection on the plane;  $\phi, \lambda$  – latitude and longitude of a point on the Earth's surface;  $X, Y$  – Cartesian coordinates of the same point on the projection;  $\phi_1, \phi_2$  are the main parallels. Calculations by (1), (2) give:

$$h = \frac{\cos \phi}{\sqrt{1 + \sin \phi_1 \sin \phi_2 - \sin \phi(\sin \phi_1 + \sin \phi_2)}}; \quad k = \frac{\sqrt{1 + \sin \phi_1 \sin \phi_2 - \sin \phi(\sin \phi_1 + \sin \phi_2)}}{\cos \phi}.$$

The graphs of the dependence of the scale factors  $h$  and  $k$  on latitude are shown in Fig. 4, distortion ellipses are shown in Fig. 5. When building, the following parameters were used:  $\phi_0 = -20^\circ, \lambda_0 = -50^\circ, \phi_1 = -30^\circ, \phi_2 = 40^\circ$ .

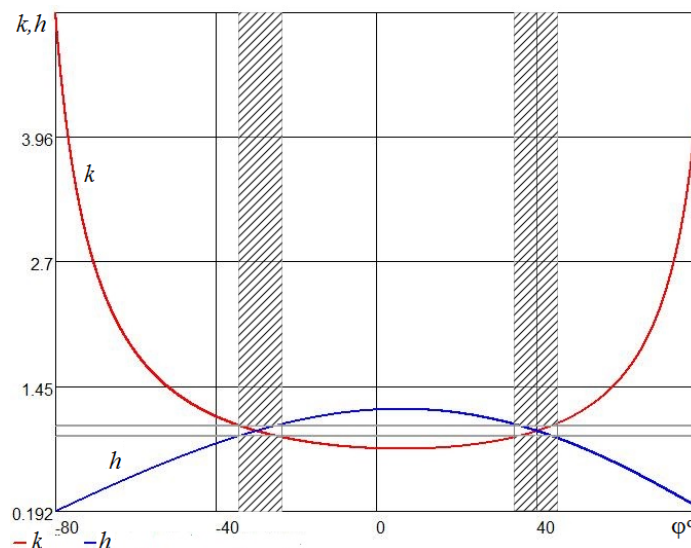


Fig. 4. Dependences of scale factors  $k(\phi)$  and  $h(\phi)$  for the Albers projection

Both scale factors are equal to 1 at  $\phi = -30^\circ$  and  $\phi = 40^\circ$ . The amount of distortion increases with distance from these values. The areas in which the scale factors differ from 1 by no more than 5% (marked with hatching in Fig.4) are small, their range is no more than  $9^\circ$ . The areal distortion coefficient will always be equal to one.



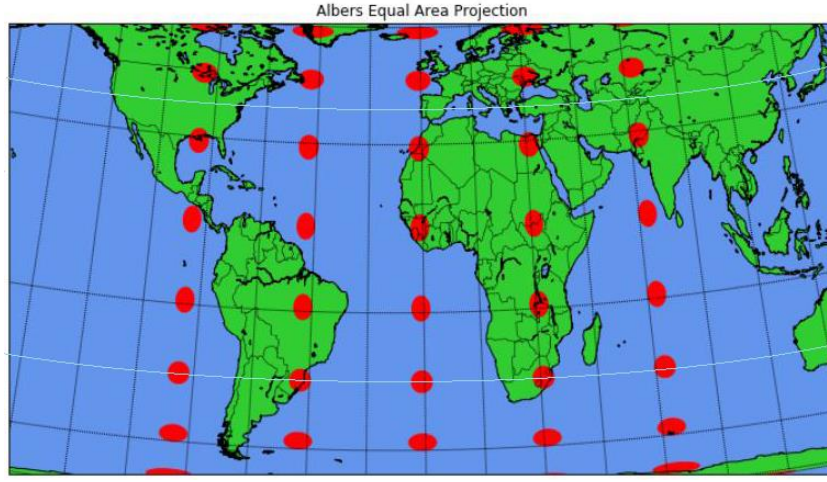


Fig. 5. The Albers projection with distortion ellipses

The two main parallels  $\phi_1 = -30^\circ$  and  $\phi_2 = 40^\circ$  are marked in Fig. 5. There are no distortions on them. However, ellipses located to the south or north stretch along the longitudes, and those ellipses that are between them stretch along the latitudes. Distortion increases with distance from the main parallel.

Now let's consider the azimuthal equidistant projection belonging to the azimuthal class. Such projections can be obtained by projecting the earth's surface onto a plane tangent to the globe. Also, azimuth projections are classified by the location of the tangent point on the globe. In the framework of this work, a polar (normal) projection is used. This means that the plane touches the globe at the pole point (in this case, the south pole).

The advantage of this projection is that it maintains the azimuth direction and distance proportions from the center point. In polar projection, all meridians are straight, distances from the pole are displayed correctly. The complexity of the projection depends on the choice of the center point. A given point on the plane is projected into Cartesian coordinates as follows:

$$\begin{aligned} X &= p \sin \theta; \\ Y &= -p \cos \theta, \end{aligned}$$

where  $\theta$  is the azimuthal angle,  $p$  is the length of the arc along the great circle between the central and projected points. In general, the relationship between coordinates  $(\theta, p)$  and latitude/longitude  $(\phi, \lambda)$  is given by the equations:

$$\begin{aligned} \cos \frac{p}{R} &= \sin \phi_1 \sin \phi + \cos \phi_1 \cos(\lambda - \lambda_0); \\ \tan \theta &= \frac{\cos \phi \sin(\lambda - \lambda_0)}{\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos(\lambda - \lambda_0)}. \end{aligned}$$

When the central point is the south pole, then  $\phi_1 = -\frac{\pi}{2}$ , and  $\lambda_0$  can be any, therefore it is most convenient to assign it a value of 0, which greatly simplifies the formulas:

$$\begin{aligned} p &= R \left( \frac{\pi}{2} + \phi \right); \\ \theta &= \lambda. \end{aligned}$$

Substituting expressions for Cartesian coordinates into expressions (1), (2), we obtain scale factors

$$\begin{aligned} h &= 1; \\ k &= \frac{\phi + \pi / 2}{\cos \phi}. \end{aligned}$$

Figure 6 shows a graph of the dependence of the scale factor  $k$  on latitude, indicating the area where the distortion does not exceed 5%. Figure 7 shows the distortion ellipses for the projection.

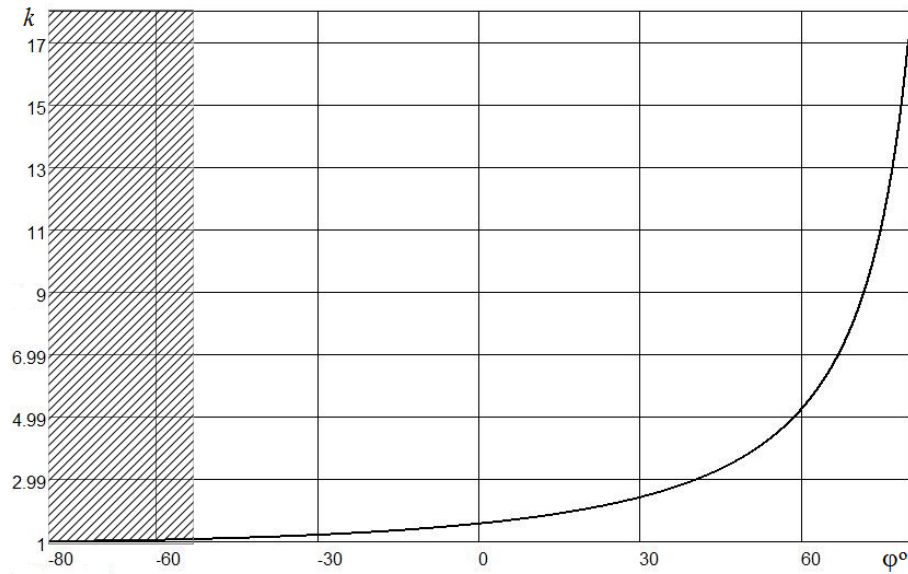


Fig. 6. Dependence of the scale factor  $k$  on latitude for the azimuthal equidistant projection

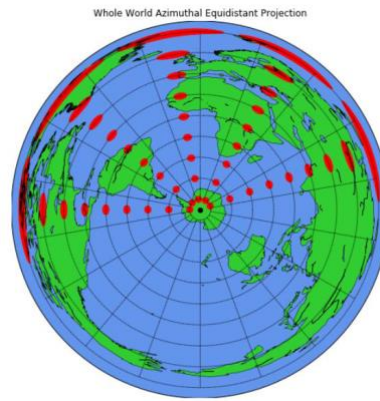


Fig. 7. Distortion ellipses in the azimuthal equidistant projection

As an example of an arbitrary projection, we use the Kavraisky projection. This is a compromise projection developed to minimize distortion across the entire surface of the globe. The Kavraisky projection is a general-purpose pseudo-cylindrical projection. Parallels are represented as straight parallel lines, and meridians are represented as curves symmetrical with respect to the average rectilinear meridian. The projection is performed according to the following formulas ( $\phi, \lambda$  - latitude and longitude of a point on the Earth's surface,  $X, Y$  are coordinates on the projection):

$$X = \frac{3\lambda}{2\pi} \sqrt{\frac{\pi^2}{3} - \phi^2};$$

$$Y = \phi.$$

According to (1), (2), we obtain the scale factors:

$$k = \frac{\sqrt{3(\pi^2 - 3\phi^2)}}{2\pi \cos \phi};$$

$$h = \sqrt{\frac{27\lambda^2 \phi^2}{\pi^2 (\pi^2 - 3\phi^2)}} + 1.$$

Figure 8 shows the dependences of the coefficients  $k$  on latitude, as well as  $h$  on longitude.

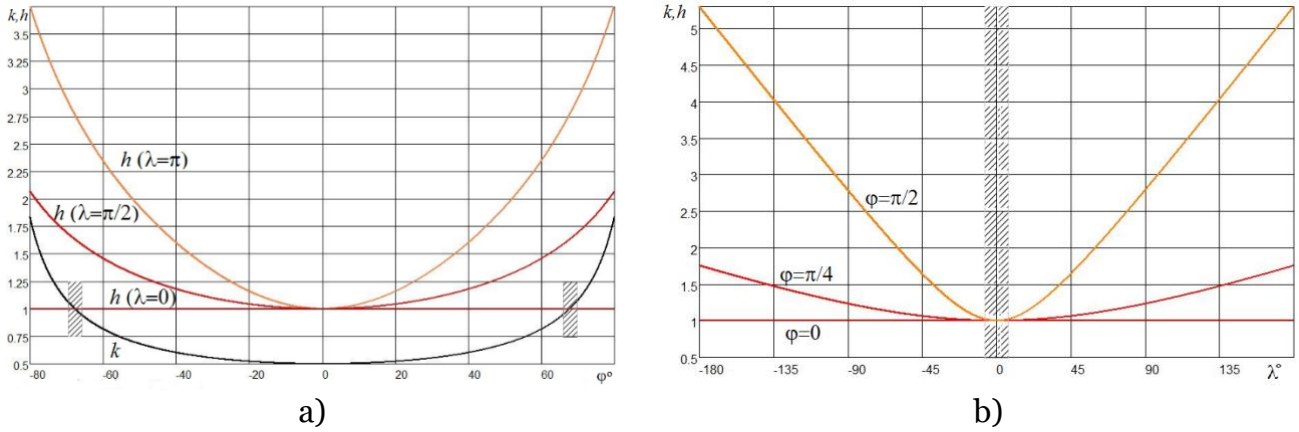


Fig. 8. Scale factors for the Kavraisky projection

The minimum distortion for this projection is reached at the point (0,0). Considering that the coefficient  $h$  simultaneously depends on  $\varphi$  and  $\lambda$ , the areas in which the distortion does not exceed 5% can only be marked near  $\lambda=0$  (areas are marked by hatching). Fig. 9 shows the display of distortion ellipsoids when using the Kavraisky projection.

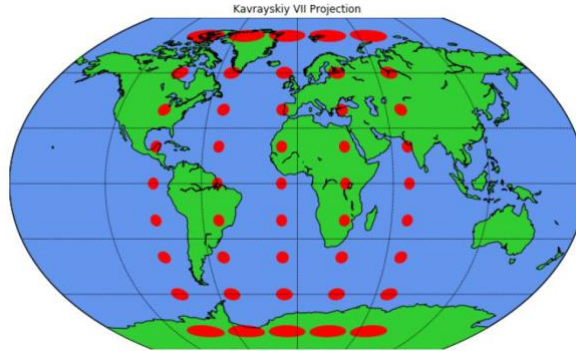


Fig. 9. Distortion ellipses in the Kavraisky projection

Since this projection has distortions in all parameters, its main area of application is geographic maps. All possible distortions are minimized here, and, therefore, the map will display the best general idea of the shape of the earth's surface.

#### 4. Methodology for estimating errors in the projection of extended objects

Estimation of display errors and distortion of the shape and size of extended objects only on the basis of partial scale factors is difficult and requires calculations.

According to the authors, it would be visual to overlay the image of an undistorted object on its display on the map. Since the undistorted figure is actually located on the ball, then if it is not an arc or a circle, questions may arise about the shape and size of the figure to overlay. The solution to the problem is seen in the use of projecting a figure onto a plane with the least distortion of size and shape. Based on the above analysis, we can suggest using the Mercator projection, assuming that the equator passes through the center of mass of the figure. Changing the latitude of the figure to  $\Delta\Phi$  can be performed by the following transformations [5]:

$$\phi_2 = \arcsin(\sin \phi_1 \cos \Delta\phi - \cos \phi_1 \sin \Delta\phi \cos \lambda_1);$$

$$\lambda_2 = \arccos\left(\frac{\sin \phi_1 \sin \Delta\phi}{\sin \lambda_1} + \cos \Delta\phi \cos \lambda_1\right),$$



where  $\lambda_1, \varphi_1$  are the longitude and latitude of the point of the original figure, and  $\lambda_2, \varphi_2$  are the coordinates of the shifted ones.

We use a parallelogram as a model figure. First of all, let's consider its mappings in the equatorial region. Fig. 10 shows a test parallelogram on a spherical Earth, Fig. 11-14 show figure using the previously discussed projections.



Fig. 10. Display of a parallelogram on the globe

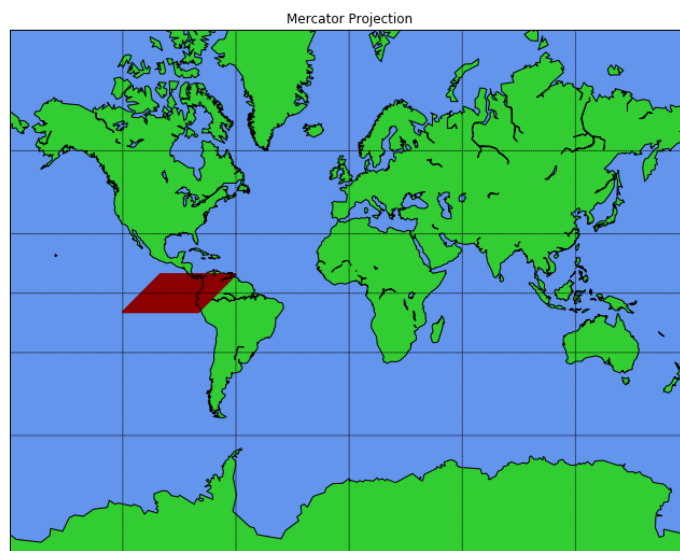


Fig. 11. Displaying a parallelogram in the Mercator projection

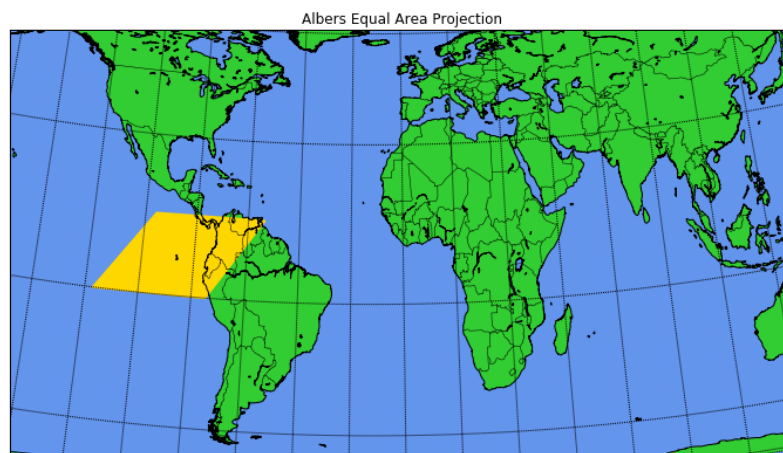


Fig. 12. Displaying a parallelogram in the Albers projection

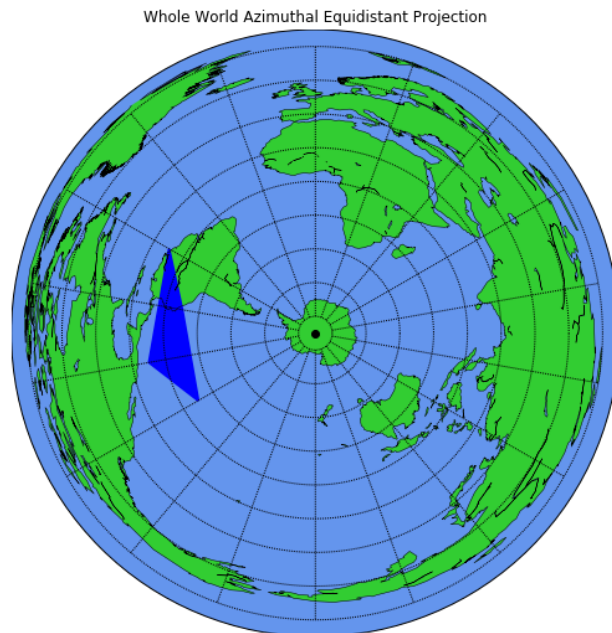


Fig. 13. Displaying a parallelogram in an azimuth equidistant projection



Fig. 14. Displaying a parallelogram in the Kavraisky projection

The above figures allow us to get a quantitative and qualitative idea of the kind of distortion for each analyzed projection. To even more clearly display the distortions occurring with the parallelogram, Figure 15 shows distorted figures on top of the original one on an equal scale.

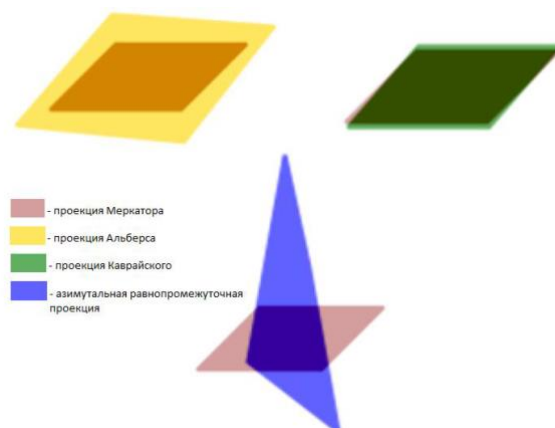


Fig. 15. Pairwise overlay of distorted parallelograms on the original

This comparison allows you to compare distortions both in magnitude and shape. The comparison, although it did not require shifting the figure to the equator, demonstrated the difference in the projection results.

The results of combining projections according to the proposed method for figures centered on the 60th parallel are shown in Fig.16-19, where the numbers are indicated:

1 - the reference figure placed on the equator in the Mercator projection;

2 - Mercator projection;

3 - Albers projection,  $\phi_1 = 20^\circ$ ,  $\phi_1 = 40^\circ$ ;

4 - azimuthal equidistant projection,  $\phi_1 = \pi / 2$ ,  $\lambda_0 = 0$ ;

5 - Kavraisky projection.

The projected figures are moved to conduct a comparative analysis. The coordinate grid is built on the assumption that the partial scale factors are equal to 1. Grid is given to estimate the amount of distortion.

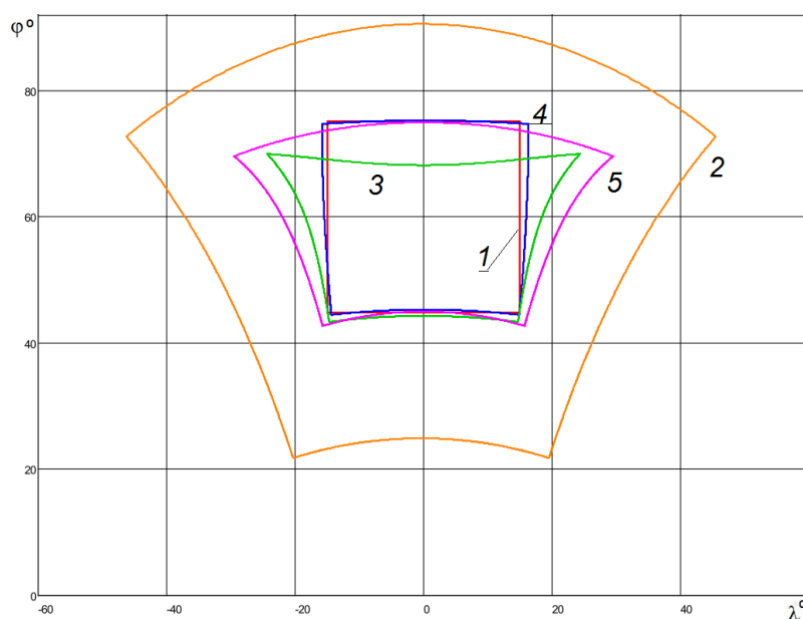


Fig. 16. Comparison of projections of a rectangular object

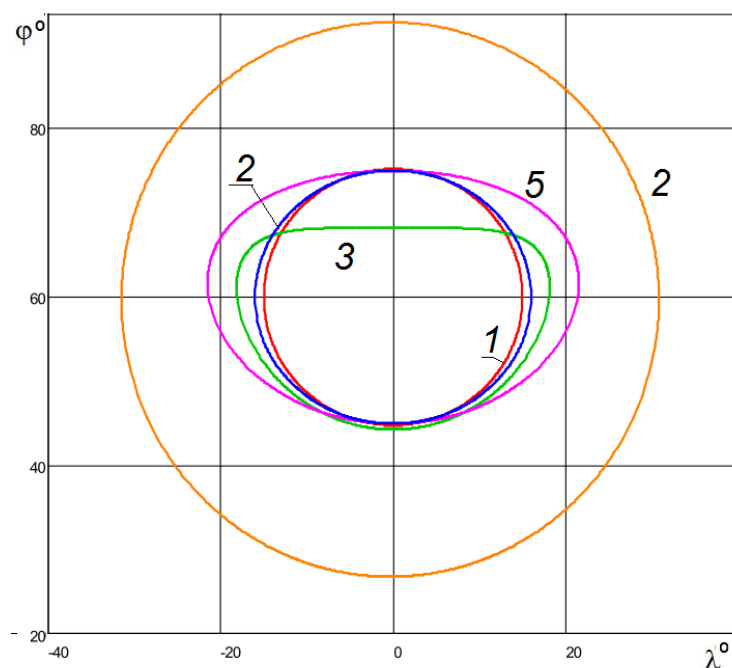


Fig. 17. Comparison of circle projections

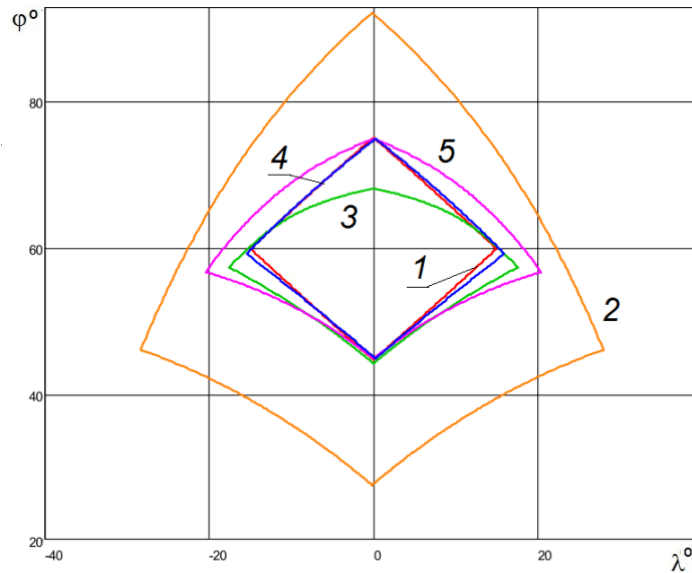


Fig. 18. Comparison of projections of a rhombus formed by arcs of a great circle

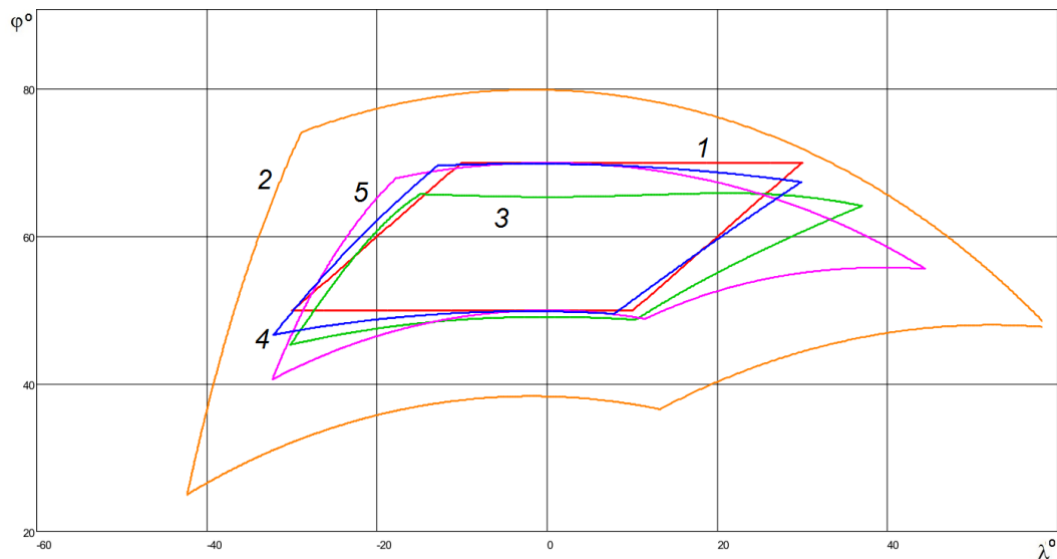


Fig. 19. Comparison of projections of a parallelogram formed by arcs of a great circle and parallels

Comparative analysis based on the combination of the reference figure and the projection clearly demonstrates the presence of significant distortions in shape and size for all considered projections. The Mercator projection showed the worst results for this location of objects. On the other hand, the reference figures are made in the Mercator projection (but in the equatorial region). For the considered examples, the best results were shown by the azimuth equidistant projection due to the successful choice of the central point. As can be seen from the comparison, the nature of the distortion varies depending on the latitude and the method of projection. This should be taken into account when working with cartographic information for the best representation of the object in each specific case.

## 5. Developed software

To obtain images on the background of the map, the authors have developed a specialized program for visualizing geometric distortions on maps. The program is written in Python using the Basemap library [6]. Basemap is a library for plotting 2D data on maps in Python. It does not perform any constructions on its own, but provides the means to transform coordinates into one of 25 different map projections (using the PROJ.4 C library). In

addition, the library is used to construct contours, images, vectors, lines or points in transformed coordinates.

The graphs of scale factors and images of figures in various projections are made in the author's program SINUS-D [7]. To prepare the data, a C ++ program has been developed that implements cartographic projection and rotation of the coordinate system.

## 6. Conclusion

When working with cartographic information, it is important to take into account the distortions introduced when projecting the earth's surface onto the map. Along with the classical means of visualization of distortions, such as ellipses of distortions, the study of partial scale factors, a comparative graphical analysis based on combining the results of projection with a projection having the shape and dimensions closest to the real object can be used. To select a reference projection, some common projections were considered. An analysis of the distortions based on the study of partial scale factors was carried out.

The implementation of the approach proposed by the authors was carried out in the software developed by them. That makes it possible to show the distortions visually and to carry out not only a qualitative, but also an evaluative quantitative analysis. The software developed by the authors makes this analysis simple and fast.

The approach to the analysis of the representation of cartographic information proposed by the authors and implemented in the programs developed by the authors can be useful primarily in studying the features of cartographic projections, but it also has practical potential in everyday use to facilitate the planning of activities and more accurate accounting and allocation of resources when it is necessary to simultaneously work with maps of different scales, in particular, in the tasks of meteorology, the use of the environment, fisheries, in emergency response, such as flooding or forest fires.

## References

1. Lebedeva O.A. Kartograficheskie projektsii [Cartographic Projections]. - M.: Novosibirsk, 2000 [in Russian].
2. Zaporozhchenko A.V. Kartograficheskie projektsii I metodika ih vybora dlya sozdaniya kart razlichnyh tipov [Cartographic Projections and Methods of Their Selection for Creating Maps of Various Types]. - M.: Panorama, 2007 [in Russian].
3. <http://kadastrua.ru/kartografiya/340-klassifikatsiya-kartograficheskikh-proektsij.html> [in Russian].
4. Bugaevsky L.M. Matematicheskaya kartografiya (Mathematical Cartography). - M.: Zlatoust, 1998 [in Russian].
5. Wentzel M.K. Sfericheskaya trigonometriya (Spherical Trigonometry). - M.: Publishing House of Geodesic and cartographic literature, 1948 [in Russian].
6. <https://matplotlib.org/basemap/index.html>
7. Ktitrov S.V. Conception and Development Experience of "SINUS-D" Software for Rapid Visualization of Dynamic Systems Simulation // Scientific Visualization, 2017, Vol.9 №3, pp.1-13.